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On the solutions to the multi-parametric Yang–Baxter equations

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To my wonderful and fair mother!

Abstract

A unified approach is applied in the consideration of the multi-parametric (colored) Yang–Baxter equations (YBE) and the usual YBE with two-parametric R-matrices, relying on the existence of the arbitrary functions in the general solutions. The colored YBE are considered with the R-matrices defined on two and three-dimensional states. We present an exhaustive study and the overall solutions for the YBE with 4×4 colored R-matrices. The established classification includes new multi-parametric free fermionic solutions. In the context of the given approach there are obtained the colored solutions to the YBE with 9×9 R-matrices having 15 non-zero elements.

1. Introduction

The Yang–Baxter equations (YBE), being formulated in the early works [1,2] (originated in [3]), have crucial role in the theory of the integrable models in low-dimensional statistical physics, quantum field theory and are the non-dividable parts of the theory of the quantum groups [4–9]. Investigations of the solutions to YBE are still actual and are involved in newer fields of the theoretical and mathematical physics.

The usual quantum YBE are formulated as the system of the equations

$$R_{ij}(u,v)R_{ik}(u,w)R_{jk}(v,w) = R_{jk}(v,w)R_{ik}(u,w)R_{ij}(u,v). \tag{1.1}$$

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Here $R_{ij}(u,v)$ is $n^2 \times n^2$ matrix, u and v are called spectral parameters. The matrix $R_{ij}(u,v)$ is considered as an operator acting on the tensor product of two n-dimensional states $V_i \otimes V_j$. The simplest and well studied solutions to YBE are the 4×4 matrices with eight non-zero entries. The symmetric solution is presented in the paper [5], which is the solution corresponding to the 2d classical statistical eight-vertex model (or the XYZ Heisenberg model). This solution has difference property $R_{ij}(u,v) = R_{ij}(u-v)$, so the R-matrix is actually one-parametric, and the corresponding YBE can be presented as

$$R_{ij}(u)R_{ik}(u+w)R_{jk}(w) = R_{jk}(w)R_{ik}(u+w)R_{ij}(u). \tag{1.2}$$

In the context of the $1+1$ quantum field theory [7,8], where the R-matrix plays the role of the scattering matrix of two particles, the YBE just ensures the factorization of the many-particle scattering matrices into the products of two-particle scattering matrices. Here the spectral parameters are simply the rapidities of the relativistic particles, and the difference property reflects the relativistic invariance of the system. Besides of the rapidity the scattering matrix can be depending also on the extra characteristics of the particles – “colors”. In the papers [13,14,19] the multi-parametric YBE are studied with colored R-matrices $R_{ij}(u;p,q)$ (the color parameters p and q are attached to the i-th and j-th spaces respectively)

$$\begin{aligned} &R_{ij}(u;p,q)R_{ik}(u+w;p,r)R_{jk}(w;q,r) \\ &\quad = R_{jk}(w;q,r)R_{ik}(u+w;p,r)R_{ij}(u;p,q). \end{aligned} \tag{1.3}$$

The corresponding solutions have the so-called free-fermionic property [11,13–17,21],

$$R_{00}^{00}R_{11}^{11} + R_{01}^{01}R_{10}^{10} - R_{01}^{10}R_{10}^{01} - R_{00}^{11}R_{11}^{00} = 0, \tag{1.4}$$

which means that the corresponding 1d quantum theories can be presented by means of the scalar free fermionic chains.

In the paper [17] there is discussed a two-parametric solution, which is just the solution to the YBE with R-matrices having two pairs of rapidities, so that for each pair the R-matrix has difference property $R(u,v;u',v') \equiv R(u-v,u'-v')$. Then YBE has the following form

$$\begin{aligned} &R_{ij}(u,u')R_{ik}(u+w,u'+w')R_{jk}(w,w') \\ &\quad = R_{jk}(w,w')R_{ik}(u+w,u'+w')R_{ij}(u,u'). \end{aligned} \tag{1.5}$$

In the recent paper [18] there are presented multi-parametric solutions, which being free-fermionic, do not correspond to the mentioned cases. The new 4×4 solutions [18] (see Subsection 4.4 therein) are formulated as two-parametric solutions, which in general case have no difference property and contain arbitrary functions.

The one of the purposes of this paper is to fill the gap, which exists in the study of the multi-parametric systems of YBE and in the hierarchy of the free-fermionic solutions for the case $n=2$. Then we show that the YBE put the restriction on the number of the possible “colors”, which can be “visible” in the solutions for the given n. For analyzing the set of YBE we propose and apply a straightforward way, formulated precisely in Section 3. We classify here the all types of the multi-parametric solutions for the matrices of the general eight-vertex kind, analyzing YBE with the 4×4 matrices. Meanwhile the whole approach could be valid also for the cases with higher-dimensional matrices, and particularly, we consider the YBE with $n=3$ too and derive a family of multi-parametric R-matrices. This set of the solutions entirely describes the situation when the R-matrices have 15 non-zero elements. The extensions of the solutions to the cases with the matrices having more non-zero elements will be presented further. In particular cases the obtained solutions correspond to the known three-parametric colored solutions, the eight-vertex model's solution and the other already observed YBE solutions both for $n = 2, 3$ [2,4–6,13–28,33].

The paper is organized in the following way. In Section 2 the YBE is presented in general multi-parametric formulation. In Subsections 2.1 and 2.2 the particular cases are discussed, corresponding to the situations $b_i = 0$ and $d_i = 0$. In Subsection 2.3 the general case is discussed. There the generalized multi-parametric version of the eight-vertex model's matrix is obtained, and we see that beside of these solutions the all remaining solutions have free-fermionic property. We are presenting the general free-fermionic solutions as matrices with two arbitrary functions (it corresponds to the four-parametric solutions) and two arbitrary constants. When one of the constants vanishes, the solution coincides with the known colored solutions [13,14]. The $sl_q(2)$-invariant solution in [18] at $q = i$ for the case $c_{1,2} = 0$ (the eigenvalues of the quadratic Casimir operator of the representations spaces on which R-matrix acts), belongs to the family of the four-parametric solutions when there are two arbitrary constants and one arbitrary function, while the other function is parameterized by exponential function in a specific way imposed by the condition of the algebra invariance. In Section 3 we present all the results and conclusions followed from the performed analysis. Therein a table is presented with the classification of the all principal types of the 4×4 solutions. In the context of the given approach in the next section we have solved multi-parametric YBE for general colored 9×9 R-matrices with 15 non-zero elements. The first part of Appendix A is devoted to the detailed discussion of the analysis of YBE for a particular case typical for the general solutions with arbitrary functions, and in the second part the main different types of the 4×4-solutions are presented in apparent matrix formulations.

2. General multiparametric solutions to YBE with 4 × 4 R-matrices

The general form of the quantum Yang–Baxter equations with multi-parameters can be presented as

$$R_{ij}(\mathbf{u},\mathbf{v})R_{ik}(\mathbf{u},\mathbf{w})R_{jk}(\mathbf{v},\mathbf{w}) = R_{jk}(\mathbf{v},\mathbf{w})R_{ik}(\mathbf{u},\mathbf{w})R_{ij}(\mathbf{u},\mathbf{v}). \tag{2.1}$$

Here under the spectral parameters $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, written in the "bold" shrift, we mean the all possible set of the parameters, which the R matrix can acquire – $\mathbf{u} = \{u_1, u_2, ...\}$, $\mathbf{v} = \{v_1, v_2, ...\}$, $\mathbf{w} = \{w_1, w_2, ...\}$. As the matrix acts on the tensor product of two vector spaces, it means that the sets of the spectral parameters are attached to the corresponding vector spaces.

Let us explore the matrices having the property of "particle number" conservation by mod(2) ($\mathcal{Z}_2$ grading symmetry). In matrix representation it means $R_{ij}^{kr} \neq 0$ if $i + j + k + r = 0 \pmod 2$. If the indexes $i, j...$ take only two values 0, 1, then R has the following 4×4 matrix form

$$R(\mathbf{u},\mathbf{w}) = \begin{pmatrix} R_{00}^{00} & 0 & 0 & R_{00}^{11} \\ 0 & R_{01}^{01} & R_{01}^{10} & 0 \\ 0 & R_{10}^{01} & R_{10}^{10} & 0 \\ R_{11}^{00} & 0 & 0 & R_{11}^{11} \end{pmatrix} \equiv \begin{pmatrix} a_1(\mathbf{u},\mathbf{w}) & 0 & 0 & d_1(\mathbf{u},\mathbf{w}) \\ 0 & b_1(\mathbf{u},\mathbf{w}) & c_1(\mathbf{u},\mathbf{w}) & 0 \\ 0 & c_2(\mathbf{u},\mathbf{w}) & b_2(\mathbf{u},\mathbf{w}) & 0 \\ d_2(\mathbf{u},\mathbf{w}) & 0 & 0 & a_2(\mathbf{u},\mathbf{w}) \end{pmatrix} \tag{2.2}$$

which is similar to the R-matrix of the eight-vertex model. We use the commonly adopted notations (2.2) for the matrix elements.

We are considering the complete and motivated cases, with $a_i \neq 0$, $c_i \neq 0$ and

$$\check{R}(\mathbf{u}, \mathbf{u}) = \mathbf{I}, \tag{2.3}$$

where $\check{R} = PR$, P is a permutation operator changing the positions of the states, and $\mathbf{I}$ is the unit operator.

The simplest equations followed from YBE (2.1) are

$$c_1(\mathbf{u}_1, \mathbf{u}_2)c_1(\mathbf{u}_2, \mathbf{u}_3)c_2(\mathbf{u}_1, \mathbf{u}_3) - c_1(\mathbf{u}_1, \mathbf{u}_3)c_2(\mathbf{u}_1, \mathbf{u}_2)c_2(\mathbf{u}_2, \mathbf{u}_3) = 0, \tag{2.4}$$

$$c_1(\mathbf{u}_1, \mathbf{u}_3)d_1(\mathbf{u}_2, \mathbf{u}_3)d_2(\mathbf{u}_1, \mathbf{u}_2) - c_2(\mathbf{u}_1, \mathbf{u}_3)d_1(\mathbf{u}_1, \mathbf{u}_2)d_2(\mathbf{u}_2, \mathbf{u}_3) = 0, \tag{2.5}$$

$$c_1(\mathbf{u}_1, \mathbf{u}_2)d_1(\mathbf{u}_2, \mathbf{u}_3)d_2(\mathbf{u}_1, \mathbf{u}_3) - c_2(\mathbf{u})d_1(\mathbf{u}_1, \mathbf{u}_3)d_2(\mathbf{u}_2, \mathbf{u}_3) = 0. \tag{2.6}$$

The general solutions to them we can parameterize by means of an arbitrary function $g(\mathbf{u})$ and an arbitrary constant d_0

$$\frac{c_2(\mathbf{u}, \mathbf{w})}{c_1(\mathbf{u}, \mathbf{w})} = \frac{g(\mathbf{u})}{g(\mathbf{w})}, \qquad \frac{d_2(\mathbf{u}, \mathbf{w})}{d_1(\mathbf{u}, \mathbf{w})} = d_0 g(\mathbf{u})g(\mathbf{w}). \tag{2.7}$$

As it is known there is a possibility to redefine the matrix elements of the matrix R_{ij} by the following transformations [9]

$$R_{n_i n_j}^{p_i p_j}(\mathbf{u}_i, \mathbf{u}_j) \quad \Rightarrow \quad R_{n_i n_j}^{p_i p_j}(\mathbf{u}_i, \mathbf{u}_j)\frac{\mathrm{f}_{n_i}(\mathbf{u}_i)\mathrm{f}_{n_j}(\mathbf{u}_j)}{\mathrm{f}_{p_i}(\mathbf{u}_i)\mathrm{f}_{p_j}(\mathbf{u}_j)}, \tag{2.8}$$

induced from the following change of the vector basis $\mathrm{e}_{n_{i,j}}$ ($n_{i,j} = 0, 1$, $p_{i,j} = 0, 1$) of the space $V_{i,j} - \mathrm{e}_{n_{i,j}} \to \mathrm{f}_{n_{i,j}}\mathrm{e}_{n_{i,j}}$. The transformations affect only the elements R_{01}^{10}, R_{10}^{01}, R_{11}^{00}, R_{00}^{11}. Taking the functions in this way $(\frac{\mathrm{f}_0(\mathbf{u})}{\mathrm{f}_1(\mathbf{u})})^2 = \sqrt{d_0}g(\mathbf{u})$, we can make the elements $c(d)_i$, $i = 1, 2$, equal. So, we can consider $c_1 = c_2$ and $d_1 = d_2$ afterwards.

We shall explore all the possible cases in detail.

We can set $c_1 = c_2 = 1$ taking into account the normalization freedom of the R-matrix. The set of the independent equations of YBE is brought in Appendix A, (A.2)–(A.7) (the conditions (A.8) ensue from the YBE for each discussed case, as it will be shown).

2.1. $a_1 = a_2$, $b_i = 0$, $i = 1, 2$

In this subsection we are demonstrating a plain case with the conditions $a_1 = a_2$, $b_i = 0$, $i = 1, 2$, for which the extra colored parameters are actually absent, and can be introduced only by taking into account the transformation freedom (2.8). From the analysis of the whole set of the equations the following general solution follows

$$a_1(\mathbf{u}, \mathbf{w}) = a_2(\mathbf{u}, \mathbf{w}) = \frac{a(\mathbf{u})a(\mathbf{w})}{1 + (d(\mathbf{u}) - d_0)d(\mathbf{w})}, \tag{2.9}$$

$$d_1(\mathbf{u}, \mathbf{w}) = d_2(\mathbf{u}, \mathbf{w}) = \frac{d(\mathbf{u}) - d(\mathbf{w})}{1 + (d(\mathbf{u}) - d_0)d(\mathbf{w})}, \tag{2.10}$$

$$a(\mathbf{u})^2 - d(\mathbf{u})^2 - 1 + d(\mathbf{u})d_0 = 0. \tag{2.11}$$

The first two equations are just the relations expressing the full functions $f_i(\mathbf{u}, \mathbf{w})$, $f = a, d$, via the elementary functions $f_i(\mathbf{u})$. We see, that the functions $a_i(\mathbf{u})$ and $d(\mathbf{u})$ are defined so, that $a_i(\mathbf{u}) = a_i(\mathbf{u}, \mathbf{u}_0)$ and $d(u) = d(\mathbf{u}, \mathbf{u}_0)$, if at the point $\mathbf{u}_0$ we have $a(\mathbf{u}_0) = 1$, $d(\mathbf{u}_0) = 0$. Note, that $\mathbf{u} = \mathbf{w}$ corresponds to the normalization point $a(\mathbf{u}, \mathbf{u}) = 1$, $d(\mathbf{u}, \mathbf{u}) = 0$. Further we can fix $\mathbf{u}_0 = \mathbf{0} = \{0, 0, ...\}$ symbolically. The last equation (2.11) is the consistency condition of the YBE. This relation is true also for the functions $a_i(\mathbf{u}, \mathbf{w})$ and $d_i(\mathbf{u}, \mathbf{w})$. Any R-matrix having the elements (2.9), (2.10) with arbitrary function $d(\mathbf{u})$ and with $a(\mathbf{u}) = \pm\sqrt{d(\mathbf{u})^2 + 1 - d(\mathbf{u})d_0}$ is YBE solution. As here we have only one arbitrary function, then the dependence from the multi-spectral parameters $\mathbf{u}$ one can encode in the argument of that function. We see, that the consistency condition implies trigonometric parameterization. If to impose the constraints $a_i(u, w) = a_i(u - w)$ and $d_i(u, w) = d_i(u - w)$, then the relations (2.9), (2.10) become functional equations on the corresponding functions and the solution is the following

$$a(u) = \frac{\sinh[u_0]}{\sinh[u + u_0]}, \qquad d(u) = \frac{\sinh[u]}{\sinh[u + u_0]}, \qquad d_0 = 2\cosh[u_0], \tag{2.12}$$

which is the solution [17] describing the XYZ-model with the coupling parameters $J_x = -J_y$, $J_z = \cosh[u_0]$.

Note, that the restriction that the dependence of the matrix elements from the spectral parameters must be in difference form $R_{ij}(u - w)$ fixes the functions $a(\mathbf{u})$, $d(\mathbf{u})$, meanwhile in general these two functions are connected by one relation (2.11). Hence one of the functions, say $a(\mathbf{u})$, is arbitrary and we can choose any parameterization for it. However by means of definite parameterization (2.12) any two-parametric solution $R_{ij}(a(\mathbf{u}), a(\mathbf{w}))$ with the conditions (2.9)–(2.11) can be brought to the actually one-parametric form $R_{ij}(u - w)$.

2.2. $d_i = 0$, $i = 1, 2$

We have $a_i(\mathbf{u}, \mathbf{u}) = 1$, $b_i(\mathbf{u}, \mathbf{u}) = 0$ from the relation (2.3). Let us again take a symbolic fixed point $\mathbf{u} = \mathbf{0}$, and set $a_i(\mathbf{u}) = a_i(\mathbf{u}, \mathbf{0})$, $b_i(\mathbf{u}) = b_i(\mathbf{u}, \mathbf{0})$. The YB equations give the following expressions for the complete functions $f_i(\mathbf{u}, \mathbf{w})$ by means of the elementary functions $f_i(\mathbf{u}) = f_i(\mathbf{u}, \mathbf{0})$

$$\begin{aligned}
a_1(\mathbf{u}, \mathbf{w}) &= b_1(\mathbf{w})b_2(\mathbf{u}) + \frac{a_1(\mathbf{u})}{a_1(\mathbf{w})}\big(1 - b_1(\mathbf{w})b_2(\mathbf{w})\big),\\
a_2(\mathbf{u}, \mathbf{w}) &= a_1(\mathbf{w})a_2(\mathbf{u}) + \frac{b_2(\mathbf{w})}{b_2(\mathbf{u})}\big(1 - a_1(\mathbf{u})a_2(\mathbf{u})\big),\\
b_1(\mathbf{u}, \mathbf{w}) &= a_2(\mathbf{w})b_1(\mathbf{u}) - a_2(\mathbf{u})b_1(\mathbf{w}),\\
b_2(\mathbf{u}, \mathbf{w}) &= a_1(\mathbf{w})b_2(\mathbf{u}) - a_1(\mathbf{u})b_2(\mathbf{w}).
\end{aligned} \tag{2.13}$$

The all relations between the functions $a_i(u)$, $b_i(u)$, coming from the YBE, can be expressed by the following constraints of familiar type

$$\frac{a_1(\mathbf{u})a_2(\mathbf{u}) + b_1(\mathbf{u})b_2(\mathbf{u}) - 1}{b_1(\mathbf{u})a_1(\mathbf{u})} = \Delta \tag{2.14}$$

$$\frac{a_2(\mathbf{u})b_2(\mathbf{u})}{a_1(\mathbf{u})b_1(\mathbf{u})} = k. \tag{2.15}$$

The relations (2.14), (2.15) in general case are not valid for the full matrix elements $f_i(\mathbf{u}, \mathbf{w})$. Here the parameters Δ, k are constants, and the second equation (2.15) must take place when $\Delta \neq 0$ (otherwise it is not necessary condition). Indeed, one could also immediately this analyzing the YBE in the common way [5], considering Eqs. (2.1) as homogeneous linear equations in respect of the functions $f_i(\mathbf{u}, \mathbf{v})$ (or either $f_i(\mathbf{u}, w)$ or $f_i(\mathbf{v}, \mathbf{w})$). Then the relations on the matrix elements are arisen from the consistency conditions of the homogeneous equations (formulated as vanishing of the determinants of the matrices composed by the corresponding coefficient functions). Taking now the YBE as homogeneous linear equations in respect of the functions depending, say on the parameters $(\mathbf{v}, \mathbf{w})$, and writing down the consistency conditions of the equations as

$$\begin{aligned}&\big(a_1(\mathbf{u},\mathbf{v})a_2(\mathbf{u},\mathbf{w})b_1(\mathbf{u},\mathbf{v})b_2(\mathbf{u},\mathbf{w}) - a_1(\mathbf{u},\mathbf{w})a_2(\mathbf{u},\mathbf{v})b_1(\mathbf{u},\mathbf{w})b_2(\mathbf{u},\mathbf{v})\big)\\ &\quad\times\big(a_1(\mathbf{u},\mathbf{v})a_2(\mathbf{u},\mathbf{v}) + b_1(\mathbf{u},\mathbf{v})b_2(\mathbf{u},\mathbf{v}) - 1\big) = 0,\end{aligned} \tag{2.16}$$

we see that when the free fermionic condition takes place (the second row in (2.16) vanishes), then it is not necessity for the vanishing of the first row.

2.2.1. $\Delta \neq 0$

In general, the solutions contain two arbitrary functions, say $a_2(\mathbf{u})$ and $b_1(\mathbf{u})$ (the other two functions, $a_1(\mathbf{u})$ and $b_2(\mathbf{u})$, can be obtained from the constraints (2.14), (2.15)) and two arbitrary parameters Δ and k

$$\begin{aligned}a_1(\mathbf{u}) &= \frac{a_2(\mathbf{u})}{a_2(\mathbf{u})^2 - \Delta a_2(\mathbf{u})b_1(\mathbf{u}) + kb_1(\mathbf{u})^2},\\ b_2(\mathbf{u}) &= \frac{kb_1(\mathbf{u})}{a_2(\mathbf{u})^2 - \Delta a_2(\mathbf{u})b_1(\mathbf{u}) + kb_1(\mathbf{u})^2}.\end{aligned} \tag{2.17}$$

As it was noted, we can encode dependence from two arbitrary functions into the arguments - two sets of independent spectral parameters ($\mathbf{u} = \{u, p\}$, $\mathbf{w} = \{w, q\}$), $f_i(\mathbf{u}, \mathbf{w}) = f_i(u, w; p, q)$. The choice of the appropriate parameterization, in which the dependence from some variables would have difference property, say $R(u, w; p, q) = R(u - w; p, q)$, brings to differential equations. Taking then $\mathbf{0} = \{0, 0\}$, we shall have $f_i(u, w; p, 0) = f_i(u - w, p)$, where $f_i(\mathbf{u}) = f_i(\mathbf{u}, 0) \equiv f_i(u, 0; p, 0) = f_i(u; p)$. Combining the relations (2.13), (2.14), (2.15), and expanding the functions $f_i(u, w; p, 0)$ near the point $w = 0$, we shall come to $f_i''(u; p) \approx f_i(u, ; p)$ (the second differential is taken over the variable u). It means that the parameterization must be taken by trigonometric functions over the variables u, w. So, the functions will have the form $f_i(u; p) = g_i(p)\sinh[u + u_i]$, with appropriate chosen functions $g_i(p)$ and constants u_i, depending on k, Δ. From another hand the desired parameterization one could obtain in this way. We can see that the functions $g(\mathbf{u})$ and $g(\mathbf{w})$, with $g(\mathbf{u})^2 = a_2(\mathbf{u})^2 - \Delta a_2(\mathbf{u})b_1(\mathbf{u}) + kb_1(\mathbf{u})^2$, are factorized in the expressions of the functions $f_i(\mathbf{u}, \mathbf{w})$. Taking $a_2(\mathbf{u}) = g(\mathbf{u})\sinh[\phi(\mathbf{u}) + \phi_0]/\sinh[\phi_0]$, $b_1(\mathbf{u}) = \sinh[\mathbf{u}]g(\mathbf{u})/(c_0 \sinh\phi_0)$, with $(\sinh[\phi_0])^2 = (\Delta^2 - 4k)/(4k)$ and $c_0 = \sqrt{k}$ (i.e. $\Delta = 2\cosh[\phi_0]c_0$), the solution can be written as

$$R(\mathbf{u},\mathbf{w}) = \begin{pmatrix} \frac{g(\mathbf{w})\sinh[\phi(\mathbf{u})-\phi(\mathbf{w})+\phi_0]}{g(\mathbf{u})\sinh[\phi_0]} & 0 & 0 & 0\\ 0 & \frac{g(\mathbf{u})g(\mathbf{w})\sinh[\phi(\mathbf{u})-\phi(\mathbf{w})]}{c_0\sinh[\phi_0]} & 1 & 0\\ 0 & 1 & \frac{c_0\sinh[\phi(\mathbf{u})-\phi(\mathbf{w})]}{g(\mathbf{u})g(\mathbf{w})\sinh[\phi_0]} & 0\\ 0 & 0 & 0 & \frac{g(\mathbf{u})\sinh[\phi(\mathbf{u})-\phi(\mathbf{w})+\phi_0]}{g(\mathbf{w})\sinh[\phi_0]}\end{pmatrix}. \tag{2.18}$$

The arguments in the previous discussion correspond to $\mathbf{u} = \{u, p\}$, $\mathbf{w} = \{w, q\}$, and $\phi(\mathbf{u}) = u$, $g(\mathbf{u}) = p$, $\phi(\mathbf{w}) = w$, $g(\mathbf{w}) = q$. If to take into account also the parameters followed from the transformations (2.8), then after the following notations

$$\frac{\mathrm{f}_1(\mathbf{u})}{\mathrm{f}_0(\mathbf{u})} \equiv t, \qquad \frac{\mathrm{f}_1(\mathbf{w})}{\mathrm{f}_0(\mathbf{w})} \equiv s, \tag{2.19}$$

we can write the $R(\mathbf{u},\mathbf{w})$-matrix as follows (now $\mathbf{u}=\{u,p,t\}$ and $\mathbf{w}=\{w,q,s\}$ and $\mathbf{0}=\{0,1,1\}$)

$$R(u-w;\,p,q;\,s/t)=\begin{pmatrix}\frac{q\sin(u-w+u_0)}{p\sin u_0} & 0 & 0 & 0\\ 0 & \frac{pq\sin(u-w)}{c_0\sin u_0} & \frac{t}{s} & 0\\ 0 & \frac{s}{t} & \frac{c_0\sin(u-w)}{pq\sin u_0} & 0\\ 0 & 0 & 0 & \frac{p\sin(u-w+u_0)}{q\sin u_0}\end{pmatrix}. \tag{2.20}$$

Of course, also another choices are possible for parameterization, say $\phi(\mathbf{u})=u$, $g(\mathbf{u})=e^{u}(p+u)$, $\frac{\mathrm{f}_1(\mathbf{u})}{\mathrm{f}_0(\mathbf{u})}=e^{\alpha u}t$, and so on. The dependence from the arbitrary constant c_0 can be eliminated by redefinition $g(\mathbf{u})\to\sqrt{c_0}g(\mathbf{u})$ ($p\to\sqrt{c_0}p$, $q\to\sqrt{c_0}q$). From the ordinary XXZ the parameterization (2.20) differs by transformations of the basis states similar to the discussed one (2.8), if to redefine the matrix elements, as it was for the solution (5.25) of the paper [17] $r_{X\_XZ}(u)\colon R_{ij}^{i'j'}\to R_{i\,\bar{j}}^{i'\,\bar{j}'}$, where $\bar{i}=\mathrm{mod}[i+1]2$ (i.e. it corresponds to the transformations (2.8) after interchanging the indexes 0 and 1).

The generalized relations similar to Eqs. (2.14) and (2.15) for the complete functions $f_i(\mathbf{u},\mathbf{w})$ are the following ones

$$\frac{a_1(\mathbf{u},w)a_2(\mathbf{u},\mathbf{w})+b_1(\mathbf{u},\mathbf{w})b_2(\mathbf{u},\mathbf{w})-1}{2\sqrt{a_1(\mathbf{u},\mathbf{w})b_1(\mathbf{u},\mathbf{w})a_2(\mathbf{u},\mathbf{w})b_2(\mathbf{u},\mathbf{w})}}=\cosh[\phi_0] \tag{2.21}$$

$$\frac{a_2(\mathbf{u},\mathbf{w})b_2(\mathbf{u},\mathbf{w})}{a_1(\mathbf{u},\mathbf{w})b_1(\mathbf{u},\mathbf{w})}=k/g(\mathbf{w})^4. \tag{2.22}$$

2.2.2. $\Delta=0$

Here the first relation in (2.14) is enough for the functions in (2.13) with arbitrary $a_i(\mathbf{u})$, $b_i(\mathbf{u})$, connected by one constraint, to be solutions to YBE. The matrix elements of $\check{R}(\mathbf{u},\mathbf{w})$, taken into account the consistency conditions, can be written as

$$a_1(\mathbf{u},\mathbf{w})=a_1(\mathbf{u})a_2(\mathbf{w})+b_1(\mathbf{w})b_2(\mathbf{u}),\qquad a_2(\mathbf{u},\mathbf{w})=a_1(\mathbf{w},\mathbf{u}) \tag{2.23}$$

$$b_1(\mathbf{u},\mathbf{w})=b_1(\mathbf{u})a_2(\mathbf{w})-b_1(\mathbf{w})a_2(\mathbf{u}),\qquad b_2(\mathbf{u},\mathbf{w})=b_2(\mathbf{u})a_1(\mathbf{w})-b_2(\mathbf{w})a_1(\mathbf{u}), \tag{2.24}$$

$$1-a_1(\mathbf{u})a_2(\mathbf{u})-b_1(\mathbf{u})b_2(\mathbf{u})=0. \tag{2.25}$$

As in the previous discussion, we can interpret the solutions in two equivalent ways. We can say, that we have three independent and arbitrary functions (e.g. a_1, a_2, b_1) in the solution, or we can say, that we have a solution with six parameters $\{a_1(u),a_2(u),b_1(u);a_1(w),a_2(w),b_1(w)\}$ (three pairs of independent parameters). *In general, the two-parametric solution with n arbitrary functions is equivalent to the* $2n$*-parametric solution*. Writing the functions in terms of the composite parameters $\mathbf{u}$, $\mathbf{w}$ we unify and extend two interpretations.

The demand that the dependence from the spectral parameters to be in difference form (with one spectral parameter) brings the solutions to the known cases, which are the XX-model's matrix ($b_i(u)=\sin u$, $a_i(u)=\cos u$) or the matrix of the XX-model in transverse magnetic field, with $b_2(u)=b_0b_1(u)=\sin u/\sin u_0$ and $(a_1(u)-a_2(u))/b_1(u)=\text{constant}\equiv 2\cos u_0$, $a_1=\sin(u+u_0)/\sin u_0$, $a_2(u)=\sin(u_0-u)/\sin u_0$. When $u_0=\pi/2$ the second case corresponds to the first one. In general, we can take as example

$$a_1(\mathbf{u}) = f(\mathbf{u}) \frac{\sin[\phi(\mathbf{u}) + \phi_0]}{\sin[\phi_0]}, \qquad b_1(\mathbf{u}) = g(\mathbf{u}) \frac{\sin[\phi(\mathbf{u})]}{\sin[\phi_0]}, \tag{2.26}$$

$$a_2(\mathbf{u}) = \frac{1}{f(\mathbf{u})} \frac{\sin[\phi_0 - \phi(\mathbf{u})]}{\sin[\phi_0]}, \qquad b_2(\mathbf{u}) = \frac{1}{g(\mathbf{u})} \frac{\sin[\phi(\mathbf{u})]}{\sin[\phi_0]}, \tag{2.27}$$

with arbitrary multiparametric functions $f(\mathbf{u})$, $g(\mathbf{u})$, $\phi(\mathbf{u})$ and arbitrary constant ϕ_0. Note, that in the unique constraint (2.25) there is no any constant, and hence ϕ_0 is not a relevant constant and can be eliminated by appropriate reparameterizations, and the same is valid also for the constant u_0 introduced below in (2.28).

With the parameterization (2.27) the independent functions are expressed by the arbitrary functions $\phi(\mathbf{u})$, $f(\mathbf{u})$, $g(\mathbf{u})$. When $f(\mathbf{u}) = 1/g(\mathbf{u})$ the $\check{R}$-matrix in terms of the argument-function $\phi(\mathbf{u})$ acquires difference property. If to set the composite arguments consisting of three parameters, $\mathbf{u} = \{u, p, \bar{p}\}$ and $\mathbf{w} = \{w, q, \bar{q}\}$, and fix the functions in this way

$$\phi(\mathbf{u}) = u, \qquad f(\mathbf{u}) = p, \qquad g(\mathbf{u}) = \bar{p},$$

we can write the matrix elements of $R(u, w; p, q; \bar{p}, \bar{q})$ as

$$\begin{aligned}
a_1(u, w; p, q; \bar{p}, \bar{q}) &= \frac{p \sinh(u + u_0) \sinh(u_0 - w)}{q (\sinh u_0)^2} + \frac{\bar{q} \sinh u \sinh w}{\bar{p} (\sinh u_0)^2}, \\
a_2(u, w; p, q; \bar{p}, \bar{q}) &= \frac{q \sinh(w + u_0) \sinh(u_0 - u)}{p (\sinh u_0)^2} + \frac{\bar{p} \sinh u \sinh w}{\bar{q} (\sinh u_0)^2}, \\
b_1(u, w; p, q; \bar{p}, \bar{q}) &= \frac{\bar{p} \sinh u \sinh(u_0 - w)}{q (\sinh u_0)^2} - \frac{\bar{q} \sinh(u_0 - u) \sinh w}{p (\sinh u_0)^2}, \\
b_2(u, w; p, q; \bar{p}, \bar{q}) &= \frac{q \sinh(w + u_0) \sinh u}{\bar{p} (\sinh u_0)^2} - \frac{p \sinh(u + u_0) \sinh w}{\bar{q} (\sinh u_0)^2}.
\end{aligned} \tag{2.28}$$

Taking into account also the parameters followed from the automorphism (2.8), and the notations (2.19), we can introduce new parameters t, s and write as well (below now $\mathbf{u} = \{u, p, \bar{p}, t\}$ and $\mathbf{w} = \{w, q, \bar{q}, s\}$) the expressions

$$c_1(\mathbf{u}, \mathbf{w}) = t/s, \qquad c_2(\mathbf{u}, \mathbf{w}) = s/t, \tag{2.29}$$

for the c_i elements of the matrix $R(u, w; p, q; \bar{p}, \bar{q}; t/s)$.

2.3. $d_i \neq 0$, $b_i \neq 0$, $i = 1, 2$

The case with the choice $d(\mathbf{u}, \mathbf{w}) \neq 0$ brings to the following relations for the full functions via the elementary ones (below $\bar{i} = i + 1 \bmod(2)$)

$$\begin{aligned}
a_i(\mathbf{u}, \mathbf{w}) &= \frac{d(\mathbf{u}, \mathbf{w})(d(\mathbf{u})[a_i(\mathbf{u}) a_i(\mathbf{w}) - b_{\bar{i}}(\mathbf{u}) b_{\bar{i}}(\mathbf{w})] + d(\mathbf{w})[a_{\bar{i}}(\mathbf{u}) a_{\bar{i}}(\mathbf{w}) - b_i(\mathbf{u}) b_i(\mathbf{w})])}{d(\mathbf{u})^2 - d(\mathbf{w})^2}, \\
b_i(\mathbf{u}, \mathbf{w}) &= \frac{a_{\bar{i}}(\mathbf{u}) b_i(\mathbf{w}) - a_{\bar{i}}(\mathbf{w}) b_i(\mathbf{u}) + d(\mathbf{u}) d(\mathbf{w})[a_i(\mathbf{u}) b_{\bar{i}}(\mathbf{w}) - a_i(\mathbf{w}) b_{\bar{i}}(\mathbf{u})]}{d(\mathbf{u})^2 d(\mathbf{w})^2 - 1}.
\end{aligned} \tag{2.30}$$

As previously, we set $f_i(\mathbf{u}, \mathbf{0}) = f_i(\mathbf{u})$, with $f = a, b, d$ and $i = 1, 2$ and $a_i(\mathbf{0}) = 1$, $b_i(\mathbf{0}) = 0$, $d(\mathbf{0}) = 0$ following from (2.3).

Analyzing the YBE with the functions (2.30) we obtain several expressions for $d(\mathbf{u}, \mathbf{w})$ by means of the elementary functions $f_i(\mathbf{u})$, $f_i(\mathbf{w})$. Consistency conditions for these expressions to be equal one to another constitute the relations between the elementary functions.

Before proceeding farther, let us reveal the nature of the constants included in the solutions by means of the derivatives of the elementary functions at the normalization point $\mathbf{0}$. For the multiparametric functions $f_i(\mathbf{w})$ we can suggest that derivation $\frac{d}{d\mathbf{w}}$ is taken along a path $\{w_i(w_0)\}$, $i=1,...,k$, if the composite parameter $\mathbf{w}=\{w_1,w_2,...,w_k\}$ is parameterized by a path parameter w_0, $w_0\in\mathcal{C}$. We imply so, $\frac{df_i(\mathbf{w})}{d\mathbf{w}}d\mathbf{w}=(\frac{dw_1}{dw_0}\partial_{w_1}+\frac{dw_2}{dw_0}\partial_{w_2}+...+\frac{dw_k}{dw_0}\partial_{w_k})f_i(\mathbf{w})dw_0$, and $f_i'(\mathbf{0})=(\frac{dw_1}{dw_0}\partial_{w_1}+\frac{dw_2}{dw_0}\partial_{w_2}+...+\frac{dw_k}{dw_0}\partial_{w_k})f_i(\mathbf{w})|_{\mathbf{w}=\mathbf{0}}$. We can see, that the constant d_0 in (2.9)–(2.11) can be expressed as $d_0=-\frac{2a'(\mathbf{0})}{d'(\mathbf{0})}$. The constants Δ and k in (2.14), (2.15) are equal to

$$\Delta=\frac{a_1'(\mathbf{0})+a_2'(\mathbf{0})}{b_1'(\mathbf{0})},\qquad k=\frac{b_2'(\mathbf{0})}{b_1'(\mathbf{0})}.$$

It means that the classification of the solutions by means of the constant parameters of the solutions can be formulated in the language of the derivatives taken at the normalization point. Particularly, we have seen for the case $d_i(\mathbf{u},\mathbf{w})=0$, that the solutions with $a_1'(\mathbf{0})+a_2'(\mathbf{0})=0$ describe free-fermionic models. We shall ascertain that this is a general property.

So, below, we shall use derivatives at the point $\mathbf{0}$ for describing the constants.

The consistency conditions, mentioned at the beginning of this subsection, contain the following general relation valid for each case

$$(\star)\quad \mathbf{2d(u)}\big[\mathbf{a_1'(0)}-\mathbf{a_2'(0)}\big]=\mathbf{d'(0)}\big(\mathbf{a_1(u)^2}-\mathbf{a_2(u)^2}-\mathbf{b_1(u)^2}+\mathbf{b_2(u)^2}\big).\tag{2.31}$$

Let us discuss different cases separately. The next part of this section is organized as follows. In the first subsection we discuss the case when the constant in the constraint (2.31) $(a_1'(\mathbf{0})-a_2'(\mathbf{0}))/d'(\mathbf{0})$ vanishes. Here we discuss in detail the two possible situations $b_1(\mathbf{u})=\pm b_2(\mathbf{u})$ $(b_1'(\mathbf{0})=\pm b_2'(\mathbf{0}))$, which after appropriate parameterizations are corresponding respectively to the solution of the XYZ model and to the two-parametric free-fermionic R-matrix. In the next subsections the solutions with non-vanishing constant $(a_1'(\mathbf{0})-a_2'(\mathbf{0}))/d'(\mathbf{0})\neq 0$ is presented. It appears that all the solutions in this case have the free fermionic property and for them $(a_1'(\mathbf{0})=-a_2'(\mathbf{0}))$ in general. In Subsection 2.3.2 we separate two cases with $b_1'(\mathbf{0})=\pm b_2'(\mathbf{0})$. One solution here can be considered as the generalization of the free-fermionic XY model's R-matrix (for which $a_i'(\mathbf{0})=0$), the second one corresponds to the known colored thee-parametric solution [13]. In Subsection 2.3.3 the general free-fermionic solutions are presented with two arbitrary functions and two non-vanishing constants.

2.3.1. $a_1'(\mathbf{0})=a_2'(\mathbf{0})$: $a_1(\mathbf{u})=a_2(\mathbf{u})$

Here we consider the situation with $a_1'(\mathbf{0})=a_2'(\mathbf{0})$. The analysis of the next equations brings to the relation $a_1(\mathbf{u})^2=a_2(\mathbf{u})^2$. This means that we must consider the case $a_1(\mathbf{u})=a_2(\mathbf{u})$, as $a_1(\mathbf{0})=a_2(\mathbf{0})=1$ (2.3). One can verify, that the situation $a_1(\mathbf{u})=-a_2(\mathbf{u})$ brings to the rather trivial solutions.

From the relation (2.31) we obtain $b_1(\mathbf{u})=\pm b_2(\mathbf{u})$. At first we discuss the case $b_1(\mathbf{u})=b_2(\mathbf{u})$.

• $b_1(\mathbf{u})=b_2(\mathbf{u})$

This choice immediately implies $a_1(\mathbf{u},\mathbf{w})=a_2(\mathbf{u},\mathbf{w})$, $b_1(\mathbf{u},\mathbf{w})=b_2(\mathbf{u},\mathbf{w})$. Let us omit by now the indexes $i=1,2$. So, in fact, this case is equivalent to the Baxter's discussion of YBE for the XYZ-model. The functions $f(\mathbf{u})$ obey to the following constraints:

$$d(\mathbf{u}) = \frac{d'(\mathbf{0})}{b'(\mathbf{0})} a(\mathbf{u}) b(\mathbf{u}), \tag{2.32}$$

$$a(\mathbf{u})^2 + b(\mathbf{u})^2 - 1 - d(\mathbf{u})^2 = \frac{2a'(\mathbf{0})}{b'(\mathbf{0})} a(\mathbf{u}) b(\mathbf{u}). \tag{2.33}$$

Similar relations take place for the functions $f(\mathbf{u}, \mathbf{w})$ too, with the same constants $\frac{d'(\mathbf{0})}{b'(\mathbf{0})} \equiv k$, $\frac{a'(\mathbf{0})}{b'(\mathbf{0})} \equiv \Delta$, as it follows from the relations (2.30). Thus we have the following expressions for the functions $f_i(\mathbf{u}, \mathbf{w})$ with one arbitrary function $a(\mathbf{u})$ and two constants, as the other functions $d(\mathbf{u})$ and $b(\mathbf{u})$ can be expressed by means of them using the relations (2.32), (2.33).

$$\begin{aligned} a(\mathbf{u}, \mathbf{w}) &= \frac{\frac{a(\mathbf{u})}{a(\mathbf{w})}(1 - b(\mathbf{w})^2) + (1 - k^2 a(\mathbf{u})^2) b(\mathbf{u}) b(\mathbf{w})}{1 - k^2 a(\mathbf{u})^2 b(\mathbf{w})^2}, \\ b(\mathbf{u}, \mathbf{w}) &= \frac{a(\mathbf{w}) b(\mathbf{u}) - a(\mathbf{u}) b(\mathbf{w})}{1 - k^2 a(\mathbf{u}) a(\mathbf{w}) b(\mathbf{u}) b(\mathbf{w})}, \\ d(\mathbf{u}, \mathbf{w}) &= k a(\mathbf{u}, \mathbf{w}) b(\mathbf{u}, \mathbf{w}). \end{aligned} \tag{2.34}$$

We can set here the usual XYZ elliptic parameterization, placing the arbitrariness of the solution on the argument function $\phi(\mathbf{u})$, and the constants ϕ_0 and $\mathbf{k}$ (elliptic module), and writing

$$a(\mathbf{u}) = \frac{\mathrm{sn}[\phi(\mathbf{u}) + \phi_0, \mathbf{k}]}{\mathrm{sn}[\phi_0, \mathbf{k}]}, \qquad b(\mathbf{u}) = \frac{\mathrm{sn}[\phi(\mathbf{u}), \mathbf{k}]}{\mathrm{sn}[\phi_0, \mathbf{k}]}, \qquad \phi(\mathbf{0}) = 0. \tag{2.35}$$

So, here actually we have only one-parameteric solution, as the functions $f_i(\mathbf{u}, \mathbf{w})$ acquire difference property by this parameterization $f(\phi(\mathbf{u}), \phi(\mathbf{w})) = f(\phi(\mathbf{u}) - \phi(\mathbf{w}))$.

• $b_2(\mathbf{u}) = -b_1(\mathbf{u})$

Now let us turn to the discussion of the case $b_2(\mathbf{u}) = -b_1(\mathbf{u}) \equiv -b(\mathbf{u})$. Here we found that $a'(\mathbf{0}) = 0$ and

$$a(\mathbf{u})^2 - b(\mathbf{u})^2 - d(\mathbf{u})^2 - 1 = 0. \tag{2.36}$$

This is a free-fermionic condition, as it immediately implies

$$a(\mathbf{u}, \mathbf{w})^2 - b(\mathbf{u}, \mathbf{w})^2 - d(\mathbf{u}, \mathbf{w})^2 - 1 = 0. \tag{2.37}$$

Also we have $b_1(\mathbf{u}, \mathbf{w}) = -b_1(\mathbf{w}, \mathbf{u}) = -b_2(\mathbf{u}, \mathbf{w})$ and $a_1(\mathbf{u}, \mathbf{w}) = a_1(\mathbf{w}, \mathbf{u}) = a_2(\mathbf{u}, \mathbf{w})$. The relation (2.36) appears to be enough for the $\check{R}(\mathbf{u}, \mathbf{w})$-matrix to satisfy the Yang–Baxter equations. So, there are two arbitrary functions in this solution. One can choose a parameterization, which will bring to the familiar solution (see, e.g. [17]). Let $1 + d(\mathbf{u})^2 = g(\mathbf{u})^2$ and $a(\mathbf{u}) = g(\mathbf{u}) \cosh[\phi(\mathbf{u})]$, $b(\mathbf{u}) = g(\mathbf{u}) \sinh[\phi(\mathbf{u})]$, then the full functions become

$$\begin{aligned} a_1\big(\phi(\mathbf{u}), \phi(\mathbf{w}); g(\mathbf{u}), g(\mathbf{w})\big) &= \frac{\cosh[\phi(\mathbf{u}) - \phi(\mathbf{w})] g(\mathbf{u}) g(\mathbf{w})}{1 + \sqrt{g(\mathbf{u})^2 - 1}\sqrt{g(\mathbf{w})^2 - 1}}, \\ b_1\big(\phi(\mathbf{u}), \phi(\mathbf{w}); g(\mathbf{u}), g(\mathbf{w})\big) &= \frac{\sinh[\phi(\mathbf{u}) - \phi(\mathbf{w})] g(\mathbf{u}) g(\mathbf{w})}{1 + \sqrt{g(\mathbf{u})^2 - 1}\sqrt{g(\mathbf{w})^2 - 1}}, \\ d\big(\phi(\mathbf{u}), \phi(\mathbf{w}); g(\mathbf{u}), g(\mathbf{w})\big) &= \frac{\sqrt{g(\mathbf{u})^2 - 1} - \sqrt{g(\mathbf{w})^2 - 1}}{1 + \sqrt{g(\mathbf{u})^2 - 1}\sqrt{g(\mathbf{w})^2 - 1}}. \end{aligned} \tag{2.38}$$

Setting

$$g(\mathbf{u}) = \frac{1}{\cos[\psi(\mathbf{u})]}, \tag{2.39}$$

we shall obtain the two-parametric solution [17] $\check{R}(\phi(\mathbf{u})-\phi(\mathbf{w}), \psi(\mathbf{u})-\psi(\mathbf{w}))$ brought in apparent matrix form in Appendix A (A.23).

2.3.2. $a_1'(\mathbf{0}) \neq a_2'(\mathbf{0})$

For the case $a_1'(\mathbf{0}) \neq a_2'(\mathbf{0})$, and hence $a_1(\mathbf{u}) \neq a_2(\mathbf{u})$, besides of the relation (2.31) also the free-fermionic property for the functions $f_i(\mathbf{u})$, $f = a, b, c, d$, is arisen

$$(\star\star) \quad \mathbf{a_1(u)a_2(u) + b_1(u)b_2(u) = 1 + d(u)^2}. \tag{2.40}$$

This relation takes place for the functions $f_i(\mathbf{u}, \mathbf{w})$ too. Analyzing the next part of the YBE we obtain the following equation

$$\big(a_1'(\mathbf{0}) + a_2'(\mathbf{0})\big)\big(a_1(\mathbf{u})b_1(\mathbf{u}) - a_2(\mathbf{u})b_2(\mathbf{u})\big) = 0. \tag{2.41}$$

If to take the relation $a_1(\mathbf{u})b_1(\mathbf{u}) = a_2(\mathbf{u})b_2(\mathbf{u})$ the next equations give that

$$\mathbf{a_1'(0) = -a_2'(0)} \tag{2.42}$$

nevertheless. One could obtain this constraint also directly from the relation (2.40) expanding it around the point $\mathbf{0}$, i.e. it is the peculiarity of the free-fermionic property providing that the condition (2.3) takes place. Thus the relation (2.41) in some sense is the analog of the relation (2.16), although all the solutions here are free-fermionic. Anyway, let us at first consider the situation with the constraint $a_1(\mathbf{u})b_1(\mathbf{u}) = a_2(\mathbf{u})b_2(\mathbf{u})$, which is the feature of the usual eight-vertex model and gives the solution with one arbitrary function and two constants.

- *The case $a_1(\mathbf{u})b_1(\mathbf{u}) = a_2(\mathbf{u})b_2(\mathbf{u})$: $b_1'(\mathbf{0}) = b_2'(\mathbf{0})$*

  The functions $f_i(\mathbf{u})$ satisfy the following relations

$$a_1(\mathbf{u})\big(b_1(\mathbf{u})^2 + a_2(\mathbf{u})^2\big) = a_2(\mathbf{u})\big(1 + d(\mathbf{u})^2\big), \tag{2.43}$$

$$f_0 b_1(\mathbf{u})\big(a_1(\mathbf{u})^2 + a_2(\mathbf{u})^2\big) = 2a_2(\mathbf{u})d(\mathbf{u}), \tag{2.44}$$

$$4x_f d(\mathbf{u})a_2(\mathbf{u})^2 = \big(a_1(\mathbf{u})^2 - a_2(\mathbf{u})^2\big)\big(a_2(\mathbf{u})^2 + b_1(\mathbf{u})^2\big). \tag{2.45}$$

The first relation is just the free-fermionic condition. The parameters x_f and f_0 are arbitrary constants and are chosen so, that $x_f = a_1'(\mathbf{0})/d_1'(\mathbf{0})$ and $f_0 = d_1'(\mathbf{0})/b_1'(\mathbf{0})$. Note that for this case $b_1'(\mathbf{0}) = b_2'(\mathbf{0})$. The solutions can be parameterized in the following form (below $x_0 = \frac{x_f}{f_0} = \frac{a_1'(\mathbf{0})}{b_1'(\mathbf{0})}$)

$$\begin{aligned}
a_1(\mathbf{u}) &= \sqrt{1 + x_0 f_x(\mathbf{u})}\,F_x(\mathbf{u}), \qquad a_2(\mathbf{u}) = \sqrt{1 - x_0 f_x(\mathbf{u})}\,F_x(\mathbf{u}) \\
b_1(\mathbf{u}) &= \frac{\sqrt{1 - x_0 f_x(\mathbf{u})}\,(1 - \sqrt{1 - f_x(\mathbf{u})^2})}{f_x(\mathbf{u})} F_x(\mathbf{u}), \\
b_2(\mathbf{u}) &= \frac{\sqrt{1 + x_0 f_x(\mathbf{u})}\,(1 - \sqrt{1 - f_x(\mathbf{u})^2})}{f_x(\mathbf{u})} F_x(\mathbf{u}) \\
d(\mathbf{u}) &= \frac{f_0 F_x(\mathbf{u})^2 (1 - \sqrt{1 - f_x(\mathbf{u})^2})}{f_x(\mathbf{u})}, \\
F_x(\mathbf{u}) &= \frac{\sqrt{\sqrt{1 - x_0^2 f_x(\mathbf{u})^2} - \sqrt{1 - (x_0^2 + f_0^2) f_x(\mathbf{u})^2}}}{f_0(1 - \sqrt{1 - f_x(\mathbf{u})^2})}.
\end{aligned} \tag{2.46}$$

We have introduced the function $f_x(\mathbf{u})$ as an independent function which has the property $f_x(\mathbf{0}) = 0$. Of course it is possible to choose different parameterizations, taking for independent function, as example the function $d(\mathbf{u})$. It could seem that here (taking into account the existence of the factors with square roots in (2.46)) it is reasonable to impose the elliptic functions, but note, that there are two arbitrary constants in the square root factors and the elliptic parameterization is significant only when one of them vanishes. Indeed, when we try to parameterize so, that the difference property to be for a pair of the parameters (u, w) from the set $\{\mathbf{u}, \mathbf{w}\}$, $f_i(u, w; ...) = f_i(u - w; ...)$, then we come to the simple elliptic parameterization with the particular case $a_1'(\mathbf{0}) = -a_2'(\mathbf{0}) = 0$, i.e. $x_0 = 0$, and $a_1(\mathbf{u}) = a_2(\mathbf{u})$, $b_1(\mathbf{u}) = b_2(\mathbf{u})$, which is the solution corresponding to the XY-model, containing one arbitrary constant.

The complete functions $f(\mathbf{u}, \mathbf{w})$ can be found from (2.30), the expression for $d(\mathbf{u}, \mathbf{w})$ is brought in Appendix A. Let us note that here the condition $a_1(\mathbf{u})b_1(\mathbf{u}) = a_2(\mathbf{u})b_2(\mathbf{u})$ is not universal, i.e. $a_1(\mathbf{u}, \mathbf{w})b_1(\mathbf{u}, \mathbf{w}) \neq a_2(\mathbf{u}, \mathbf{w})b_2(\mathbf{u}, \mathbf{w})$ at general. In particular, we have $a_1(\mathbf{0}, \mathbf{u}) = a_2(\mathbf{u})$, $a_2(\mathbf{0}, \mathbf{u}) = a_1(\mathbf{u})$, $b_1(\mathbf{0}, \mathbf{u}) = -b_1(\mathbf{u})$, $b_2(\mathbf{0}, \mathbf{u}) = -b_2(\mathbf{u})$ and $a_1(\mathbf{0}, \mathbf{u})b_2(\mathbf{0}, \mathbf{u}) = a_2(\mathbf{0}, \mathbf{u})b_1(\mathbf{0}, \mathbf{u})$. The free fermionic condition in that context is universal.

One can prove (after some tangled calculations), that the functions above satisfy to the whole set of YBE, although they can be obtained only by considering the particular cases of them, when one of the composite arguments has been taken to be $\mathbf{0}$. This is a natural result, *as the point* $\mathbf{0}$ *is chosen arbitrarily, we could take any point* $\mathbf{w}_0$ *for defining the elementary functions* $f_i(\mathbf{u}) = f_i(\mathbf{u}, \mathbf{w}_0)$*, and then the primary values* $f_i(\mathbf{w}_0)$ *for them will be fixed from the normalization condition* (2.3). Also the free-fermionic condition can be proved for the entire functions $f_i(\mathbf{u}, \mathbf{w})$, $f = a, b, c, d$. Another parameterization would be given in Appendix A, with a short description of a receipt how to solve YBE. So, for this particular case we have the solutions with one arbitrary function and two arbitrary constants, and in general this is a two-parametric solution.

The solution discussed above is the special case of the general solutions with the property $a_1'(\mathbf{0}) = -a_2'(\mathbf{0})$. And in general from the overall analysis of the YBE we come to the next important constraint

$$(\star\star\star) \quad \mathbf{d}(\mathbf{u})\big(\mathbf{b}_1'(\mathbf{0}) + \mathbf{b}_2'(\mathbf{0})\big) = \mathbf{2d}'(\mathbf{0})\big(\mathbf{a}_1(\mathbf{u})\mathbf{b}_2(\mathbf{u}) + \mathbf{a}_2(\mathbf{u})\mathbf{b}_1(\mathbf{u})\big). \tag{2.47}$$

This relation together with (2.31) and (2.40) is enough for $f_i(\mathbf{u})$ to satisfy the whole set of YBE. The corresponding general solutions will be discussed in Subsection 2.3.3. Here we should like to investigate separately the case with the vanishing constant $(b_1'(\mathbf{0}) + b_2'(\mathbf{0}))/d'(0) = 0$ (i.e. $a_1(\mathbf{u})b_2(\mathbf{u}) + a_2(\mathbf{u})b_1(\mathbf{u}) = 0$) (2.47) which as we shall see, corresponds to the elliptic three-parametric solution [24].

• *The case* $a_1(\mathbf{u})b_2(\mathbf{u}) + a_2(\mathbf{u})b_1(\mathbf{u}) = 0$: $b_1'(\mathbf{0}) = -b_2'(\mathbf{0})$

There are three relations on the five functions $a_1(\mathbf{u})$, $a_2(\mathbf{u})$, $b_1(\mathbf{u})$, $b_2(\mathbf{u})$, $d(\mathbf{u})$, and hence the solutions contain two arbitrary functions (four-parametric solution) and one arbitrary constant ($x_f = \frac{a_1'(\mathbf{0})}{d'(\mathbf{0})} = \frac{a_1'(\mathbf{0}) - a_2'(\mathbf{0})}{2d'(\mathbf{0})}$, which comes from (2.31)). From the study of the relations it appears that this is just the "colored" solution presented in the works [13,14], provided that we require difference property for one of the pairs of the parameters. Let us describe this solution.

Fixing the arbitrary functions as $d(\mathbf{u})$ and $f_z(\mathbf{u}) = b_2(\mathbf{u})/a_2(\mathbf{u})$ we find for the remaining functions (below two auxiliary functions $g_x(\mathbf{u})$ and $d_f(\mathbf{u})$ are introduced and instead of $d(\mathbf{u})$ the function $d_f(\mathbf{u})$ could be considered as elementary function)

$$a_1(\mathbf{u}) = g_x(\mathbf{u})\sqrt{\frac{1 + d(\mathbf{u})^2}{g_x(\mathbf{u})(1 - f_z(\mathbf{u})^2)}}, \qquad b_1(\mathbf{u}) = -g_x(\mathbf{u}) f_z(\mathbf{u})\sqrt{\frac{1 + d(\mathbf{u})^2}{g_x(\mathbf{u})(1 - f_z(\mathbf{u})^2)}},$$

$$a_2(\mathbf{u}) = \sqrt{\frac{1+d(\mathbf{u})^2}{g_x(\mathbf{u})(1-f_z(\mathbf{u})^2)}},$$

$$g_x(\mathbf{u}) = x_f d_f(\mathbf{u}) \pm \sqrt{1+x_f^2 d_f(\mathbf{u})^2}, \quad d_f(\mathbf{u}) = \frac{2d(\mathbf{u})}{1+d(\mathbf{u})^2}. \tag{2.48}$$

The matrix elements now look like as

$$d(\mathbf{u},\mathbf{w}) = 2\frac{[d(\mathbf{u})^2 - d(\mathbf{w})^2]}{[1+d(\mathbf{u})^2][1+d(\mathbf{w})^2](d_f(\mathbf{u})g_x(\mathbf{w}) + d_f(\mathbf{w})g_x(\mathbf{u}) - 2x_f d_f(\mathbf{u})d_f(\mathbf{w}))}, \tag{2.49}$$

$$\begin{aligned} a_1(\mathbf{u},\mathbf{w}) = {} & \frac{2[[1+d(\mathbf{u})^2][1+d(\mathbf{w})^2][1-f_z(\mathbf{u})^2][1-f_z(\mathbf{w})^2]]^{-\frac{1}{2}}}{[d_f(\mathbf{u})g_x(\mathbf{w}) + d_f(\mathbf{w})g_x(\mathbf{u}) - 2x_f d_f(\mathbf{u})d_f(\mathbf{w})]} \\ & \times \left(d(\mathbf{u})\sqrt{g_x(\mathbf{u})g_x(\mathbf{w})} + d(\mathbf{w})\frac{1}{\sqrt{g_x(\mathbf{u})g_x(\mathbf{w})}} \right. \\ & \left. - f_z(\mathbf{u})f_z(\mathbf{w})\left[\frac{d(\mathbf{u})}{\sqrt{g_x(\mathbf{u})g_x(\mathbf{w})}} + d(\mathbf{w})\sqrt{g_x(\mathbf{u})g_x(\mathbf{w})}\right]\right), \end{aligned} \tag{2.50}$$

$$\begin{aligned} b_1(\mathbf{u},\mathbf{w}) = {} & \sqrt{\frac{[1+d(\mathbf{u})^2][1+d(\mathbf{w})^2]}{[1-f_z(\mathbf{u})^2][1-f_z(\mathbf{w})^2][-1+d(\mathbf{u})^2d(\mathbf{w})^2]}} \\ & \times \left(f_z(\mathbf{u})\left[\sqrt{\frac{g_x(\mathbf{u})}{g_x(\mathbf{w})}} - d(\mathbf{u})d(\mathbf{w})\sqrt{\frac{g_x(\mathbf{w})}{g_x(\mathbf{u})}}\right] \right. \\ & \left. - f_z(\mathbf{w})\left[\sqrt{\frac{g_x(\mathbf{w})}{g_x(\mathbf{u})}} - d(\mathbf{u})d(\mathbf{w})\sqrt{\frac{g_x(\mathbf{u})}{g_x(\mathbf{w})}}\right]\right), \end{aligned} \tag{2.51}$$

$$\begin{aligned} b_2(\mathbf{u},\mathbf{w}) = {} & \sqrt{\frac{[1+d(\mathbf{u})^2][1+d(\mathbf{w})^2]}{[1-f_z(\mathbf{u})^2][1-f_z(\mathbf{w})^2][-1+d(\mathbf{u})^2d(\mathbf{w})^2]}} \\ & \times \left(f_z(\mathbf{w})\left[\sqrt{\frac{g_x(\mathbf{u})}{g_x(\mathbf{w})}} - d(\mathbf{u})d(\mathbf{w})\sqrt{\frac{g_x(\mathbf{w})}{g_x(\mathbf{u})}}\right] \right. \\ & \left. - f_z(\mathbf{u})\left[\sqrt{\frac{g_x(\mathbf{w})}{g_x(\mathbf{u})}} - d(\mathbf{u})d(\mathbf{w})\sqrt{\frac{g_x(\mathbf{u})}{g_x(\mathbf{w})}}\right]\right), \end{aligned} \tag{2.52}$$

$$a_2(\mathbf{u},\mathbf{w}) = a_1(\mathbf{w},\mathbf{u}). \tag{2.53}$$

The function $d(\mathbf{u},\mathbf{w})$ depends only on the elementary functions $d_f(\mathbf{u})$ and $d_f(\mathbf{w})$. If to demand that $d(\mathbf{u},\mathbf{w}) = d(\mathbf{u}-\mathbf{w})$ then we shall have the following differential equation on the function $d_f(\mathbf{u})$

$$d'_f(\mathbf{u}) = d'_f(\mathbf{0})\sqrt{1-d_f(\mathbf{u})^2}\sqrt{1+x_f^2 d_f(\mathbf{u})^2}, \tag{2.54}$$

the solution of which is the Jacobi's elliptic function $\mathrm{sn}[\mathbf{u},k]$ with $k^2 = -x_f^2$. This corresponds to the colored parameterization presented in [14], if to take the composite parameters in the matrix $R(\mathbf{u},\mathbf{w})$ to be $\mathbf{u} = \{u,p\}$, $\mathbf{w} = \{v,q\}$, and to fix the next arbitrary functions as $f_z(\mathbf{u}) = p$, $f_z(\mathbf{w}) = q$. Thus one can check that the whole matrix has difference property in respect of the variables u and w: $R(\mathbf{u},\mathbf{w}) = R(u,w,p,q) = R(u-w,p,q)$. The full matrix elements are given in Appendix A. The case with $x_f = 0$ has been discussed in Subsection 2.3.1.

2.3.3. The general free-fermionic case with $a'(\mathbf{0}) = -a_2'(\mathbf{0})$

And in general case, at the condition $a'(\mathbf{0}) = -a_2'(\mathbf{0})$, the solution contains two arbitrary constants and two arbitrary functions and corresponds to the four parametric solution. The equations on the elementary functions are the already presented three relations ($\star$, $\star\star$, $\star\star\star$) (2.31), (2.40), (2.47), which are collected together below (the constants are denoted as before $x_f = \frac{a_1'(\mathbf{0})-a_2'(\mathbf{0})}{2d'(\mathbf{0})}$, $f_0 = \frac{2d'(\mathbf{0})}{b_1'(\mathbf{0})+b_2'(\mathbf{0})}$)

$$4x_f d(\mathbf{u}) = \big(a_1(\mathbf{u})^2 - a_2(\mathbf{u})^2 - b_1(\mathbf{u})^2 + b_2(\mathbf{u})^2\big), \tag{2.55}$$

$$2d(\mathbf{u}) = f_0\big(a_2(\mathbf{u})b_1(\mathbf{u}) + a_1(\mathbf{u})b_2(\mathbf{u})\big), \tag{2.56}$$

$$a_1(\mathbf{u})a_2(\mathbf{u}) + b_1(\mathbf{u})b_2(\mathbf{u}) - 1 - d(\mathbf{u})^2 = 0. \tag{2.57}$$

The function $d(\mathbf{u},\mathbf{w})$ looks similar to the expression of the previous case, only the constant parameter x_f^2 is changed into $x_f^2 - f_0^{-2}$, where $f_0^{-1} = \bar{x}_0$

$$d(\mathbf{u},\mathbf{w}) = \big\{\pm 2\big[d(\mathbf{u})^2 - d(\mathbf{w})^2\big]\big\}\Big\{\big[1+d(\mathbf{u})^2\big]\big[1+d(\mathbf{w})^2\big]\Big(d_f(\mathbf{u})\sqrt{1+\big(x_f^2-\bar{x}_0^2\big)d_f(\mathbf{w})^2} + d_f(\mathbf{w})\sqrt{1+\big(x_f^2-\bar{x}_0^2\big)d_f(\mathbf{u})^2}\Big)\Big\}^{-1}. \tag{2.58}$$

This means that the constraint $d(\mathbf{u},\mathbf{w}) = d(\mathbf{u}-\mathbf{w})$ leads to the Jacobi's elliptic function $\mathrm{sn}[\mathbf{u},k]$ with $k^2 = \bar{x}_0^2 - x_f^2$. If to request also that the remained functions have the difference property, we shall come to a particular case, with the following differential relation for the next arbitrary function chosen as $f_g(\mathbf{u}) = b_1(\mathbf{u})/a_1(\mathbf{u})$: $f_g'(\mathbf{u}) = f_g'(\mathbf{0})\frac{\sqrt{1-d_f(\mathbf{u})^2(1-f_g(\mathbf{u})^2)}}{x_f d_f(\mathbf{u})+\sqrt{1-(\bar{x}_0^2-x_f^2)d_f(\mathbf{u})^2}}$, $f_g'(\mathbf{0}) = 1/2\bar{x}_0 d_f'(\mathbf{0})$. When $\bar{x}_0 \neq 0$ ($b_1'(\mathbf{0}) + b_2'(\mathbf{0}) \neq 0$), then we shall have the non-homogeneous free fermionic solution of the paper [17] (corresponding to the 2d Ising model, the matrix form of which is presented in Appendix A, at the particular limit [10]), with the requirement $b_1 = b_2$ and the parameters $k^2 = \bar{x}_0^2 - x_f^2$ and $\mathrm{dn}[u_0,k] = x_f/\bar{x}_0$.

At the general case the solution can be represented by the following matrix elements (the expression of the element $d(\mathbf{u},\mathbf{w})$ is written above)

$$a_1(\mathbf{u},\mathbf{w}) = \frac{2\sqrt{[1-\bar{x}_0 f_g(\mathbf{u})d_f(\mathbf{u})][1-\bar{x}_0 f_g(\mathbf{w})d_f(\mathbf{w})]}}{\sqrt{g_x(\mathbf{u})g_x(\mathbf{w})[1+d(\mathbf{u})^2][1+d(\mathbf{w})^2][1-f_g(\mathbf{u})^2][1-f_g(\mathbf{w})^2]}} \times \frac{(d(\mathbf{u})[g_x(\mathbf{u})g_x(\mathbf{w}) - f_p(\mathbf{u})f_p(\mathbf{w})] + d(\mathbf{w})[1 - f_g(\mathbf{u})f_g(\mathbf{w})g_x(\mathbf{u})g_x(\mathbf{w})])}{(d_f(\mathbf{u})G_x(\mathbf{w}) + d_f(\mathbf{w})G_x(\mathbf{u}) - 2x_f d_f(\mathbf{u})d_f(\mathbf{w}))}, \tag{2.59}$$

$$b_1(\mathbf{u},\mathbf{w}) = a_2(\mathbf{u})a_2(\mathbf{w}) \times \frac{g_x(\mathbf{w})f_g(\mathbf{w}) - g_x(\mathbf{u})f_g(\mathbf{u}) + d(\mathbf{u})d(\mathbf{w})[g_x(\mathbf{u})f_p(\mathbf{w}) - g_x(\mathbf{w})f_p(\mathbf{u})]}{-1 + d(\mathbf{u})^2 d(\mathbf{w})^2}, \tag{2.60}$$

$$b_2(\mathbf{u},\mathbf{w}) = a_2(\mathbf{u})a_2(\mathbf{w}) \times \frac{g_x(\mathbf{u})f_p(\mathbf{w}) - g_x(\mathbf{w})f_p(\mathbf{u}) + d(\mathbf{u})d(\mathbf{w})[g_x(\mathbf{w})f_g(\mathbf{w}) - g_x(\mathbf{u})f_g(\mathbf{u})]}{-1 + d(\mathbf{u})^2 d(\mathbf{w})^2}, \tag{2.61}$$

$$a_2(\mathbf{u},\mathbf{w}) = a_1(\mathbf{w},\mathbf{u}). \tag{2.62}$$

Here we have two arbitrary and independent functions $d(\mathbf{u})$ and $f_g(\mathbf{u})$ and two constant $\bar{x}_0$ and x_f, the other functions are expressed by them through the relations

$$g_x(\mathbf{u}) = \frac{G_x(\mathbf{u})}{1 - \bar{x}_0 f_g(\mathbf{u}) d_f(\mathbf{u})}, \qquad G_x(\mathbf{u}) = x_f d_f(\mathbf{u}) \pm \sqrt{1 - \left(\bar{x}_0^2 - x_f^2\right) d_f(\mathbf{u})^2}, \tag{2.63}$$

$$f_p(\mathbf{u}) = \frac{-\bar{x}_0 d_f(\mathbf{u}) + f_g(\mathbf{u})}{-1 + \bar{x}_0 f_g(\mathbf{u}) d_f(\mathbf{u})}, \qquad d_f(\mathbf{u}) = \frac{2d(\mathbf{u})}{1 + d(\mathbf{u})^2}, \tag{2.64}$$

$$a_2(\mathbf{u}) = \sqrt{\frac{[1 + d(\mathbf{u})^2][1 - \bar{x}_0 f_g(\mathbf{u}) d_f(\mathbf{u})]}{[1 - f_g(\mathbf{u})^2] g_x(\mathbf{u})}}. \tag{2.65}$$

In general, we can write the R-matrix as

$$R_{ij}\big(d(\mathbf{u}), d(\mathbf{w}); f_g(\mathbf{u}), f_g(\mathbf{w})\big) = R_{ij}(u, w; p, q), \tag{2.66}$$

where there are done the following notations

$$f_g(\mathbf{u}) = p, \qquad f_g(\mathbf{w}) = q, \tag{2.67}$$

and we have used instead of the composite arguments the usual spectral parameters $\mathbf{u} = \{u, p\}$, with $d(\mathbf{u}) = d(u)$, $f_g(\mathbf{u}) = p$.

The $sl_q(2)$-invariant solution in the paper [18] containing one arbitrary function $h(u, \varepsilon)$, two parameters $\varepsilon_{1,2}$ (the characteristics of the cyclic irreps of the $sl_q(2)$ algebra at $q = i$) and two arbitrary constants (see Subsection 4.4 in the mentioned work) belongs to the discussed here case (2.66) – the functions $f_g(\mathbf{u})$, $d(\mathbf{u})$ are dependent from the functions $h(u, \varepsilon)$ and e^{ε} in a specific nonlinear way (it can be obtained just by comparing the matrix elements R_{ij}^{kr}), which is given in Appendix A.

3. Main conclusions and outlook

We can summarize the results and conclusions following from the analysis performed in the Section 2 regarding the 4×4 solutions of the YBE in the following interrelated points (statements).

- Yang–Baxter equations define the number of the arbitrary colored parameters (independent functions), on which the R-matrices can be dependent. For the XYZ-type 4×4 multi-parametric R-matrices, the maximal number of the relevant parameters is six (or eight, if we take into account also the parameters connected with the automorphisms (2.8)).
- As it was said in the Introduction the scattering matrices of the relativistic particles in $1 + 1$ theory depends on the difference of the rapidities and the YB equations, which in this case constitute the factorization behavior of the multi-particle scattering matrices [7–9], actually depend only on two spectral parameters. We have shown, that the YBE has the mentioned "relativity" property, even if/when the matrices $R(\mathbf{u}, \mathbf{w}) \neq R(\mathbf{u} - \mathbf{w})$, in the following sense. When we obtain the solutions to YBE (2.1) for the fixed values of one of the composite parameters, e.g. $\mathbf{w} = \mathbf{0}$, with defined initial conditions followed from (2.3), the solution is valid also for arbitrary $\mathbf{w}$.
- Starting from the usual YBE with two parametric $R(u, w)$ we can obtain all the multi-parametric solutions, as when in the solution we get the arbitrary, non-fixed function, we can regard it as another arbitrary parameter.

Table 1
Classification of the R_{22}-solutions.

The 4×4 solutions to YBE with eight non-zero matrix elements, with primary condition $R(\mathbf{u}, \mathbf{u}) = I$ classified by means of the constant parameters $f_i'(\mathbf{0})$. Three columns are overlapped at $f_i'(\mathbf{0}) = 0$, $f = a, b$.
The subsequent particular cases and the special cases of the solutions with difference property.
The constants are denoted as $x_f = \frac{a_1'(\mathbf{0})}{d'(\mathbf{0})}$, $\bar{x}_0 = \frac{b_1'(\mathbf{0})+b_2'(\mathbf{0})}{2d'(\mathbf{0})}$, $x_0 = \frac{x_f}{\bar{x}_0}$

$a_1'(\mathbf{0}) = a_2'(\mathbf{0})$		$a_1'(\mathbf{0}) = -a_2'(\mathbf{0})$
$a_1(\mathbf{u}) = a_2(\mathbf{u})$ & $b_1(\mathbf{u}) = b_2(\mathbf{u})$ generalized XYZ-type solutions with one arbitrary function and two constants:	$a_1(\mathbf{u}) = a_2(\mathbf{u})$ & $b_1(\mathbf{u}) = -b_2(\mathbf{u})$ free-fermionic solutions with two arbitrary functions	$a_1(\mathbf{u}) \neq a_2(\mathbf{u})$ free-fermionic solutions with two arbitrary functions and two constants $R(p, q, u, w)$
the complete set of the independent equations on the functions $f_1(\mathbf{u})$		*the complete set of the independent equations on the functions* $f_i(\mathbf{u})$
$\frac{a_1(\mathbf{u})^2+b_1(\mathbf{u})^2-1-d(\mathbf{u})^2}{2a_1(\mathbf{u})b_1(\mathbf{u})} = x_0$ $\bar{x}_0 d(\mathbf{u}) = a_1(\mathbf{u})b_1(\mathbf{u})$	$a_1(\mathbf{u})^2 - b_1(\mathbf{u})^2 = 1 + d(\mathbf{u})^2$	$a_1(\mathbf{u})a_2(\mathbf{u}) + b_1(\mathbf{u})b_2(\mathbf{u}) = 1 + d(\mathbf{u})^2$ $d(\mathbf{u}) = \frac{a_1(\mathbf{u})^2-a_2(\mathbf{u})^2-b_1(\mathbf{u})^2+b_2(\mathbf{u})^2}{4x_f}$ $d(\mathbf{u}) = \frac{a_1(\mathbf{u})b_2(u)+a_2(\mathbf{u})b_1(\mathbf{u})}{\bar{x}_0}$
the special cases of the solutions with difference property		*particular cases*
$R_{XYZ}(u - w)$ (2.35), (A.22)	$R(u_1 - w_1, u_2 - w_2)$ (2.39), (A.23)	$b_1'(\mathbf{0}) = -b_2'(\mathbf{0})$ (2.48), (A.25); $b_1'(\mathbf{0}) = b_2'(\mathbf{0})$ (2.46)

The case with six non-zero matrix elements: $d(\mathbf{u}, \mathbf{w}) = 0$

$a_1'(\mathbf{0}) = a_2'(\mathbf{0})$	$a_1'(\mathbf{0}) = -a_2'(\mathbf{0})$
independent equations on the functions $f_i(\mathbf{u})$ (2.17)	*independent equations on the functions* $f_i(\mathbf{u})$ (2.25)
generalized XXZ-type solutions with two arbitrary functions and two constants: $R(u - w; p, q)$ (2.20)	free-fermionic solution with three arbitrary functions: $R(u, w; p, q, \bar{p}, \bar{q})$ (2.28)

- The difference property of the dependence of the R-matrices from the spectral parameters is natural property in the scattering theory, when the matrix plays the role of the scattering matrix of two particles, and the spectral parameters are simply the rapidities (u, w). When the particles have additional extra symmetries (are "colored"), the R-matrix in respect of the parameters, which describe their extra characteristics ("colored" parameters p, q) may have no difference property, as it was in the well known solutions [13,14,19]. The performed analysis shows, that the number of the independent extra characteristics, which can be shown in the R-matrix, is restricted (maximally two kind of "colored" parameters, if we do not take into account the automorphism considered in the first part of the paper), conditioned by the number of the arbitrary functions in the YBE solutions. In the cases, when we have no difference property for any pair of the parameters, but there are some special values of the constants, at which for a pair of the parameters the difference property can be recovered, we may consider again such parameters as "rapidities".
- Now let us turn to the constants existing in the solutions. We see, that there can be utmost two arbitrary constants, which are arisen naturally in the solutions. In the discussions they are noted by k, Δ or x_0, f_0 (we are neglecting the constants, which can be introduced via the arbitrary functions, e.g. as in (2.28), and which do not play any role in the classification of the solutions). When all the eight matrix elements are non-zero, then the classification of the solutions leads to definite relations on the elementary functions and constants formulated by means of the derivatives given at the normalization point. A principle one is $a_1'(\mathbf{0}) = \pm a_2'(\mathbf{0})$. For the "plus" sign, this relation implies $a_1(\mathbf{u}) = a_2(\mathbf{u})$ and $b_1(\mathbf{u}) = \pm b_2(\mathbf{u})$. For the "minus" sign we have $a_1(\mathbf{u}) \neq a_2(\mathbf{u})$ and free-fermionic conditions. The algebraic equations on the

functions $f_i(\mathbf{u})$, $f = a, b, d$, $i = 1, 2$ for both cases are three and are dependent on the constants $(b_1'(\mathbf{0}) + b_2'(\mathbf{0}))/d'(\mathbf{0})$ and $a_1'(\mathbf{0})/d'(\mathbf{0})$.

In Table 1 we collect together the obtained results on the principal types of the 4×4 solutions. The cases with eight and six non-zero entries are considered separately, as just by taking $d(\mathbf{u}) \to 0$ we could not recover all the multi-parametric solutions.

Note, that one can choose parameterizations so, that $f_i'(\mathbf{0}) = 0$ for all the functions f_i, but the constants do not vanish, as they must not depend on the parameterization, and then one can use instead the higher derivatives at that point. From this observation it could seem, that the classification in terms of the derivatives of the functions at the given point is dependent from the parameterization, but as the relations, which we use for classification for the solutions are correct relations for *any and each* parameterization, this classification is justified and must be understood in that sense.

3.1. General statements

In usual, solving the system of YBE we take into account, that in respect of the functions f_i with any of the arguments $(\mathbf{u}, \mathbf{w})$, $(\mathbf{u}, \mathbf{v})$ or $(\mathbf{v}, \mathbf{w})$, the YB equations form system of homogeneous linear equations. And it immediately gives consistency conditions for the solutions following from the requirement of vanishing of the determinants of the matrices formed by means of the appropriate coefficients, as it was for the symmetric eight-vertex model [5]. For large matrices, with considerable amount of different matrix elements, such determinants in general can be not so easily factorizable and hence not so much informative about the nature of the solutions. We shall adhere to another way for solving YBE – a rather simple algebraic way, described below, and implemented in the analysis for the case $n = 2$ performed in the previous sections.

Let us do the notations

$$\mathcal{K}_{ijk}[\mathbf{u}, \mathbf{v}, \mathbf{w}] \equiv R_{ij}(\mathbf{u}, \mathbf{v}) R_{ik}(\mathbf{u}, \mathbf{w}) R_{jk}(\mathbf{v}, \mathbf{w}) - R_{jk}(\mathbf{v}, \mathbf{w}) R_{ik}(\mathbf{u}, \mathbf{w}) R_{ij}(\mathbf{u}, \mathbf{v}), \tag{3.1}$$

YB equations now read as $\mathcal{K}_{ijk}[\mathbf{u}, \mathbf{v}, \mathbf{w}] = 0$. We shall consider the matrices with the property (2.3). Let us recall once again, that the parameters $\mathbf{u}$, $\mathbf{w}$ are synthesized, joint parameters with meaning of the sets of the parameters $\{u\}$, $\{w\}$, connected with the corresponding states. The matrix elements at the point $(\mathbf{u}, \mathbf{0})$, where $\mathbf{0}$ symbolically denotes a normalization point, we regard as elementary functions $f_i(\mathbf{u}, \mathbf{0}) = f_i(\mathbf{u})$. Solving YBE, we consider the following scheme.

- We fix $f_i(\mathbf{u}, \mathbf{0}) = f_i(\mathbf{u})$, with the initial conditions $f_i(\mathbf{0})$, followed from (2.3) and consider the equations $\mathcal{K}_{ijk}[\mathbf{u}, \mathbf{w}, \mathbf{0}] = 0^*$.
- We express the two-composite parametric functions in terms of the elementary functions by solving the equations (*)
  – $f_i(\mathbf{u}, \mathbf{w}) = \mathcal{F}_i[\{f_j(\mathbf{u}), f_j(\mathbf{w})\}]^{**}$.
- We find the all set of the equations put on the elementary functions by the YBE (*) and also ($\mathcal{K}_{ijk}[\mathbf{u}, \mathbf{0}, \mathbf{w}] = 0$, $\mathcal{K}_{ijk}[\mathbf{0}, \mathbf{u}, \mathbf{w}] = 0$), after inserting there the complete functions $\mathcal{F}_i$ (**)
  – $\mathcal{G}_k[\{f_j(\mathbf{u}), f_j(\mathbf{w})\}] = 0^{***}$.

The number of the independent equations $\mathcal{G}_k = 0$ defines the number of the arbitrary elementary functions which can be in solutions, or, in another words, the number of the independent parameters – possible colors. As the point $\mathbf{0}$ is chosen arbitrarily, the solutions must satisfy also to the complete equations $\mathcal{K}_{ijk}[\mathbf{u},\mathbf{v},\mathbf{w}]$. For the all obtained in this paper solutions it is verified by direct calculations. The sets $\mathcal{G}_k$ for the 4×4-matrices are presented apparently in the Table 1 for each case.

4. Colored 9 x 9 solutions to YBE

The simplest known 9×9 solution is the sl_2-invariant solution of the form $R(u,w)=P+(u-w)I$, P is the permutation operator and I is the unit matrix [22–24]. We shall consider here the simplest colored generalization of this solution, with the non-vanishing elements R_{ij}^{ij} and R_{ij}^{ji}, where the indexes $i, j\ldots$ take the values $1,\ 2,\ 3$.

$$R_{33}(\mathbf{u},\mathbf{w}) = \begin{pmatrix} a_1(\mathbf{u},\mathbf{w}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_1(\mathbf{u},\mathbf{w}) & 0 & \bar{c}_1(\mathbf{u},\mathbf{w}) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_3(\mathbf{u},\mathbf{w}) & 0 & 0 & 0 & \bar{c}_3(\mathbf{u},\mathbf{w}) & 0 & 0 \\ 0 & c_1(\mathbf{u},\mathbf{w}) & 0 & \bar{b}_1(\mathbf{u},\mathbf{w}) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_2(\mathbf{u},\mathbf{w}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_2(\mathbf{u},\mathbf{w}) & 0 & \bar{c}_2(\mathbf{u},\mathbf{w}) & 0 \\ 0 & 0 & c_3(\mathbf{u},\mathbf{w}) & 0 & 0 & 0 & \bar{b}_3(\mathbf{u},\mathbf{w}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_2(\mathbf{u},\mathbf{w}) & 0 & \bar{b}_2(\mathbf{u},\mathbf{w}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3(\mathbf{u},\mathbf{w}) \end{pmatrix} \tag{4.2}$$

We study the cases, with $a_i\neq 0$, $c_i\neq 0$ and with the condition (2.3). The simplest equations followed from YBE (2.1) are

$$c_i(\mathbf{u}_1,\mathbf{u}_2)c_i(\mathbf{u}_2,\mathbf{u}_3)\bar{c}_i(\mathbf{u}_1,\mathbf{u}_3)-c_i(\mathbf{u}_1,\mathbf{u}_3)\bar{c}_i(\mathbf{u}_1,\mathbf{u}_2)\bar{c}_i(\mathbf{u}_2,\mathbf{u}_3)=0, \tag{4.3}$$

for $i=1,2,3$. Then we have the following equations

$$c_2(\mathbf{u}_2,\mathbf{u}_3)\big(b_1(\mathbf{u}_1,\mathbf{u}_2)b_3(\mathbf{u}_1,\mathbf{u}_3)-b_1(\mathbf{u}_1,\mathbf{u}_3)b_3(\mathbf{u}_1,\mathbf{u}_2)\big)=0, \tag{4.4}$$

$$c_3(\mathbf{u}_2,\mathbf{u}_3)\big(\bar{b}_1(\mathbf{u}_1,\mathbf{u}_2)b_2(\mathbf{u}_1,\mathbf{u}_3)-\bar{b}_1(\mathbf{u}_1,\mathbf{u}_3)b_2(\mathbf{u}_1,\mathbf{u}_2)\big)=0, \tag{4.5}$$

$$c_1(\mathbf{u}_2,\mathbf{u}_3)\big(\bar{b}_3(\mathbf{u}_1,\mathbf{u}_2)\bar{b}_2(\mathbf{u}_1,\mathbf{u}_3)-\bar{b}_3(\mathbf{u}_1,\mathbf{u}_3)\bar{b}_2(\mathbf{u}_1,\mathbf{u}_2)\big)=0, \tag{4.6}$$

with similar equations when instead of c_i the functions $\bar{c}_i$ stand. The next simple equations are

$$c_1(\mathbf{u}_1,\mathbf{u}_2)\big(b_3(\mathbf{u}_2,\mathbf{u}_3)b_2(\mathbf{u}_1,\mathbf{u}_3)-b_3(\mathbf{u}_1,\mathbf{u}_3)b_2(\mathbf{u}_2,\mathbf{u}_3)\big)=0, \tag{4.7}$$

$$\bar{c}_2(\mathbf{u}_1,\mathbf{u}_2)\big(\bar{b}_1(\mathbf{u}_2,\mathbf{u}_3)\bar{b}_3(\mathbf{u}_1,\mathbf{u}_3)-\bar{b}_1(\mathbf{u}_1,\mathbf{u}_3)\bar{b}_3(\mathbf{u}_2,\mathbf{u}_3)\big)=0, \tag{4.8}$$

$$c_3(\mathbf{u}_1,\mathbf{u}_2)\big(b_1(\mathbf{u}_2,\mathbf{u}_3)\bar{b}_2(\mathbf{u}_1,\mathbf{u}_3)-b_1(\mathbf{u}_1,\mathbf{u}_3)\bar{b}_2(\mathbf{u}_2,\mathbf{u}_3)\big)=0. \tag{4.9}$$

The general solutions to the above equations we can parameterize by means of the elementary functions $f_i(\mathbf{u})\equiv f_i(\mathbf{u},\mathbf{0})$, according to the previous discussions, and the constants in the solutions we shall parameterize by the derivatives of the elementary functions taken at the point $\mathbf{u}=\mathbf{0}$. The relations which follow from the above equations (4.4)–(4.9), supposing that $R_{ij}^{ji}\neq 0$, when $i\neq j$, can be presented as

$$\frac{\bar{c}_i(\mathbf{u},\mathbf{w})}{c_i(\mathbf{u},\mathbf{w})}=\frac{\bar{c}_i(\mathbf{u})c_i(\mathbf{w})}{c_i(\mathbf{u})\bar{c}_i(\mathbf{w})},\qquad \frac{\bar{b}_1(\mathbf{u},\mathbf{w})}{b_2(\mathbf{u},\mathbf{w})}=\frac{\bar{b}_1(\mathbf{u})}{b_2(\mathbf{u})},$$
$$\frac{\bar{b}_2(\mathbf{u},\mathbf{w})}{\bar{b}_3(\mathbf{u},\mathbf{w})}=\frac{\bar{b}_2(\mathbf{u})}{\bar{b}_3(\mathbf{u})},\qquad \frac{b_3(\mathbf{u},\mathbf{w})}{b_1(\mathbf{u},\mathbf{w})}=\frac{b_3(\mathbf{u})}{b_1(\mathbf{u})}, \tag{4.10}$$

$$\bar{b}_2(\mathbf{u})=\frac{\bar{b}'_2(\mathbf{0})}{b'_1(\mathbf{0})}b_1(\mathbf{u}),\qquad b_3(\mathbf{u})=\frac{b'_3(\mathbf{0})}{b'_2(\mathbf{0})}b_2(\mathbf{u}),\qquad \bar{b}_3(\mathbf{u})=\frac{\bar{b}'_3(\mathbf{0})}{\bar{b}'_1(\mathbf{0})}\bar{b}_1(\mathbf{u}). \tag{4.11}$$

Via the transformations (2.8), now with $n_{i,j}=1,2,3$, $p_{i,j}=1,2,3$, which affect only the elements R_{ij}^{ji}, $i\neq j$, we can make two pairs of the symmetric matrix elements identical one to another. Taking the functions in this way $(\frac{\mathrm{f}_2(\mathbf{u})}{\mathrm{f}_1(\mathbf{u})})^2=\alpha_1\frac{\bar{c}_1(\mathbf{u})}{c_1(\mathbf{u})}$, $(\frac{\mathrm{f}_3(\mathbf{u})}{\mathrm{f}_1(\mathbf{u})})^2=\alpha_3\frac{\bar{c}_3(\mathbf{u})}{c_3(\mathbf{u})}$ the following elements become equal: $\bar{c}_i(\mathbf{u})=c_i(\mathbf{u})$, $i=1,3$. For $i=2$ after this choice that equality does not take place in general, as the fraction $(\frac{\mathrm{f}_2(\mathbf{u})}{\mathrm{f}_3(\mathbf{u})})$ is fixed by the previous relations, and by such basis transformations we could make $\bar{c}_2=c_2$, only if $c_2\approx c_1c_3$. So, we shall consider in the following $\bar{c}_1=c_1$, $\bar{c}_3=c_3$, $\bar{c}_2\neq c_2$. Of course there is an arbitrariness in our choice, as we could take to be equal the other two pairs ($i=2,3$ or $i=1,2$) as well and consider the situations with $\bar{c}_1\neq c_1$ or $\bar{c}_3\neq c_3$.

As any solution to YBE is defined up to a multiplicative function, we can fix also $a_1(\mathbf{u},\mathbf{w})=1$. So, initially we have 12 elementary functions $f_i(\mathbf{u})$. Then the analysis of the whole set of YBE by means of the scheme defined in Section 3 shows, that after definition of the functions $\mathcal{F}_i$ (**), the remaining equations give nine constraints $\mathcal{G}_i$ (***) on the functions $f_i(\mathbf{u})$, including the three relations (4.11). We shall not present the large expressions of the constraints $\mathcal{G}_i$ and will present immediately the solutions. So, there are three independent functions in the solutions. We take for distinctness the functions $c_i(\mathbf{u})$ as arbitrary ones. The result, after doing the following successive redefinitions $c_2(\mathbf{u})\equiv f(\mathbf{u})c_1(\mathbf{u})c_3(\mathbf{u})$, $R_{33}(\mathbf{u},\mathbf{w})\to[f(\mathbf{u})+x_f(1-f(\mathbf{w}))]R_{33}(\mathbf{u},\mathbf{w})$, $c_1(\mathbf{u})=\frac{\mathbf{c}_1(\mathbf{u})\sqrt{x_f+f(\mathbf{u})(1-x_f)}}{f(\mathbf{u})}$ and $c_3(\mathbf{u})=\frac{\mathbf{c}_3(\mathbf{u})\sqrt{x_f+f(\mathbf{u})(1-x_f)}}{f(\mathbf{u})}$, can be presented as follows

$$a_1(\mathbf{u},\mathbf{w})=f(\mathbf{u})+x_f\big(1-f(\mathbf{w})\big),$$
$$a_2(\mathbf{u},\mathbf{w})=\frac{\mathbf{c}_1(\mathbf{u})^2}{\mathbf{c}_1(\mathbf{w})^2}\left(\frac{(\alpha+1)[f(\mathbf{u})+x_f(1-f(\mathbf{w}))]+(1-\alpha)[f(\mathbf{w})+x_f(1-f(\mathbf{u}))]}{2}\right),$$
$$a_3(\mathbf{u},\mathbf{w})=\frac{\mathbf{c}_3(\mathbf{u})^2}{\mathbf{c}_3(\mathbf{w})^2}\left(\frac{(\bar{\alpha}+1)[f(\mathbf{u})+x_f(1-f(\mathbf{w}))]+(1-\bar{\alpha})[f(\mathbf{w})+x_f(1-f(\mathbf{u}))]}{2}\right),$$
$$c_1(\mathbf{u},\mathbf{w})=\frac{\mathbf{c}_1(\mathbf{u})\sqrt{x_f+f(\mathbf{u})(1-x_f)}\sqrt{x_f+f(\mathbf{w})(1-x_f)}}{\mathbf{c}_1(\mathbf{w})},$$
$$c_3(\mathbf{u},\mathbf{w})=\frac{\mathbf{c}_3(\mathbf{u})\sqrt{x_f+f(\mathbf{u})(1-x_f)}\sqrt{x_f+f(\mathbf{w})(1-x_f)}}{\mathbf{c}_3(\mathbf{w})},$$
$$c_2(\mathbf{u},\mathbf{w})=\frac{\mathbf{c}_1(\mathbf{u})\mathbf{c}_3(\mathbf{u})[x_f+f(\mathbf{u})(1-x_f)]}{\mathbf{c}_1(\mathbf{w})\mathbf{c}_3(\mathbf{w})},$$
$$\bar{c}_2(\mathbf{u},\mathbf{w})=\frac{\mathbf{c}_1(\mathbf{u})\mathbf{c}_3(\mathbf{u})[x_f+f(\mathbf{w})(1-x_f)]}{\mathbf{c}_1(\mathbf{w})\mathbf{c}_3(\mathbf{w})},$$
$$b_1(\mathbf{u},\mathbf{w})=\mathbf{c}_1(\mathbf{u})^2\big(f(\mathbf{u})-f(\mathbf{w})\big),\qquad b_3(\mathbf{u},\mathbf{w})=x_f\mathbf{c}_3(\mathbf{u})^2\big(f(\mathbf{u})-f(\mathbf{w})\big),$$
$$\bar{b}_1(\mathbf{u},\mathbf{w})=x_f\frac{f(\mathbf{u})-f(\mathbf{w})}{\mathbf{c}_1(\mathbf{w})^2},\qquad b_2(\mathbf{u},\mathbf{w})=x_f\frac{\gamma\,\mathbf{c}_3(\mathbf{u})^2[f(\mathbf{u})-f(\mathbf{w})]}{\mathbf{c}_1(\mathbf{w})^2},$$

$$\bar{b}_3(\mathbf{u},\mathbf{w}) = \frac{f(\mathbf{u}) - f(\mathbf{w})}{\mathbf{c}_3(\mathbf{w})^2}, \qquad \bar{b}_2(\mathbf{u},\mathbf{w}) = \frac{\mathbf{c}_1(\mathbf{u})^2[f(\mathbf{u}) - f(\mathbf{w})]}{\gamma\, \mathbf{c}_3(\mathbf{w})^2}. \tag{4.12}$$

The numbers α, $\bar{\alpha}$ can have only the values 1 and -1. When the constant $x_f = 1$ (i.e. $\bar{c}_2(\mathbf{u},\mathbf{w}) = c_2(\mathbf{u},\mathbf{w})$), then with respect to the function $f(\mathbf{u})$ the solution acquires the difference property $R_{33}(f(\mathbf{u}), f(\mathbf{w}); \mathbf{c}_i(\mathbf{u}), \mathbf{c}_i(\mathbf{w})) = R_{33}(f(\mathbf{u}) - f(\mathbf{w}); \mathbf{c}_i(\mathbf{u}), \mathbf{c}_i(\mathbf{w}))$. If to take $x_f = 1$, $\gamma = 1$, $\alpha = \bar{\alpha} = 1$ and $\mathbf{c}_1(\mathbf{u}) = \mathbf{c}_3(\mathbf{u}) = 1$, $f(\mathbf{u}) = 1 + u$, we shall recover the known R_{33}-matrix, which has the form $I + (u - w)P$ [22–24]. And similarly, at the same particular homogeneous case ($x_f = 1$) the other rational limits of the solutions (4.12) can coincide with the corresponding rational solutions in the recent paper [27], where the authors considered matrices with more non-vanishing elements, constrained with definite symmetry relations on the matrix elements. The $N = 3$ case of the general $U(1)^{N-1}$-symmetric R-matrices, discussed in [28], can coincide with the matrix (4.12) after some symmetry transformations, as one of the authors of [28] has kindly checked.[1] If to require that difference property takes place then the matrix (4.12) would be equivalent to the Perk–Schultz solution for the three-state case [25,26] (we are thankful to prof. J. Perk for drawing our attention to this point), where the function $f(u)$ is fixed by trigonometric function $(e^u - x_f)/(1 - x_f)$. Then setting $x_f = e^\eta$, and with appropriate choice of the remaining functions and constants, the solution in (21) of the encyclopedia article [26] can be recovered. The case $x_f = 1$ corresponds to the rational limit.

Here the general solution has actually one important arbitrary constant: $x_f = \frac{c_1'(\mathbf{0}) - \bar{c}_2'(\mathbf{0}) + c_3'(\mathbf{0})}{c_1'(\mathbf{0}) - c_2'(\mathbf{0}) + c_3'(\mathbf{0})}$, one rapidity parameter, the role of which plays the function $f(\mathbf{u})$, and two type of colors, described by functions $p = \mathbf{c}_1(\mathbf{u})$, $q = \mathbf{c}_1(\mathbf{w})$ and $\bar{p} = \mathbf{c}_3(\mathbf{u})$, $\bar{q} = \mathbf{c}_3(\mathbf{w})$. We can denote the obtained matrices as

$$R_{33}^{\alpha\bar{\alpha}}(u, w; p, q; \bar{p}, \bar{q}), \quad \alpha, \bar{\alpha} = \pm 1. \tag{4.13}$$

Also the irrelevant colors $s = \frac{\mathrm{f}_2(\mathbf{u})}{\mathrm{f}_1(\mathbf{u})}$, $\bar{s} = \frac{\mathrm{f}_2(\mathbf{w})}{\mathrm{f}_1(\mathbf{w})}$, $t = \frac{\mathrm{f}_3(\mathbf{u})}{\mathrm{f}_1(\mathbf{u})}$ and $\bar{t} = \frac{\mathrm{f}_3(\mathbf{w})}{\mathrm{f}_1(\mathbf{w})}$ one could take into account, if to recall the transformation freedom connected with the basis renormalization (2.8). Then the matrix elements will be changed in this way $c_1(\mathbf{u},\mathbf{w}) \to [s/\bar{s}]c_1(\mathbf{u},\mathbf{w})$, $\bar{c}_1(\mathbf{u}, w) \to [\bar{s}/s]c_1(\mathbf{u},\mathbf{w})$, $c_3(\mathbf{u},\mathbf{w}) \to [t/\bar{t}]c_3(\mathbf{u},\mathbf{w})$, $c_2(\mathbf{u},\mathbf{w}) \to [st/(\bar{s}\bar{t})]c_2(\mathbf{u},\mathbf{w})$ and $\bar{c}_2(\mathbf{u},\mathbf{w}) \to [\bar{s}\bar{t}/(st)]\bar{c}_2(\mathbf{u},\mathbf{w})$, forming the “eight-parametric” matrix $R_{33}^{\alpha\bar{\alpha}}(u, w; p, q; \bar{p}, \bar{q}; s/\bar{s}; t/\bar{t})$.

In the scheme of the Quantum Inverse Scattering Problem [4] the structure of the one-dimensional quantum spin Hamiltonian corresponding to the R_{33}-matrix can be seen presenting the $\check{R}_{33}$ in the operator form, preliminary doing the notations

$$e^{\pm} = \frac{S^z(S^z \pm 1)}{2}, \qquad e^0 = I - \left(S^z\right)^2, \qquad S^{ik} = S^i S^k, \quad i,k = +,-,z, \tag{4.14}$$

where I is the unite operator and $S^{\pm} = \sqrt{2}J^{\pm}$, $S^z = J^z$. J^i are the normalized 3×3 spin-1 generators of the sl_2-algebra, $J^+ = \frac{1}{\sqrt{2}}\begin{pmatrix} & 1 & \\ & & 1 \\ & & \end{pmatrix}$, $J^- = \frac{1}{\sqrt{2}}\begin{pmatrix} & & \\ 1 & & \\ & 1 & \end{pmatrix}$ and $J^z = \begin{pmatrix} 1 & & \\ & & \\ & & -1 \end{pmatrix}$. Then the R-matrix has the general structure (below we are omitting the arguments $\mathbf{u}, \mathbf{w}$ of the functions)

$$\begin{aligned}
\check{R}_{33} = {} & a_1 e^+ \otimes e^+ + a_2 e^0 \otimes e^0 + a_3 e^- \otimes e^- \\
& + b_1 S^{z+} \otimes S^{-z} + \bar{b}_1 S^{-z} \otimes S^{z+} + b_2 S^{+z} \otimes S^{z-} + \bar{b}_2 S^{z-} \otimes S^{+z} + b_3 S^{++} \otimes S^{--} \\
& + \bar{b}_3 S^{--} \otimes S^{++} + c_1 e^0 \otimes e^+ + \bar{c}_1 e^+ \otimes e^0 + c_2 e^- \otimes e^0 + \bar{c}_2 e^0 \otimes e^- \\
& + c_3 e^- \otimes e^+ + \bar{c}_3 e^+ \otimes e^-.
\end{aligned} \tag{4.15}$$

[1] R. Pimenta, e-mail correspondence.

Let us adopt a convention that we have one kind spectral parameter $\mathbf{u} = u$, $f(\mathbf{u}) = u + 1$ and two arbitrary functions $\mathbf{c}_{1,3}(u)$. Then expanding by the variable w the $\check{R}_{33}(u, w)$-matrix near the point u, the linear terms in the expansion will correspond to the local cell terms $H_{i,i+1}$ of the respective spin-Hamiltonian with the interactions between the nearest-neighbors spins (acting on the spaces $V_i \otimes V_{i+1}$): $\check{R}_{33} \approx I \otimes I + (w - u)H_{i,i+1}$. For simplicity we can take $\mathbf{c}_i$ to be constant: $\mathbf{c}_1(u) = e^{\varepsilon_1/2}$ and $\mathbf{c}_3(u) = e^{\varepsilon_3/2}$, ε_i are numbers. The extension for the general family of the Hamiltonian operators with arbitrary functions $\mathbf{c}_{1,3}(u)$ would be obvious.

$$\begin{aligned}\check{R}_{33}(u, w) = I \otimes I + \frac{w - u}{1 + u(1 - x_f)} P_{\alpha,\bar{\alpha},\gamma,\varepsilon_i} + \frac{w - u}{1 + u(1 - x_f)} (x_f - 1) \bar{P}_{\alpha,\bar{\alpha},\gamma,\varepsilon_i} \\ + O(w - u),\end{aligned} \tag{4.16}$$

$$\begin{aligned}P_{\alpha,\bar{\alpha},\gamma,\varepsilon_i} = {} & e^+ \otimes e^+ + \alpha\, e^0 \otimes e^0 + \bar{\alpha}\, e^- \otimes e^- + e^{\varepsilon_1} S^{z+} \otimes S^{-z} + e^{-\varepsilon_1} S^{-z} \otimes S^{z+} \\ & + \left(e^{\varepsilon_3-\varepsilon_1}\right)\gamma S^{+z} \otimes S^{z-} + \frac{e^{\varepsilon_1-\varepsilon_3}}{\gamma} S^{z-} \otimes S^{+z} \\ & + e^{\varepsilon_3} S^{++} \otimes S^{--} + e^{-\varepsilon_3} S^{--} \otimes S^{++},\end{aligned} \tag{4.17}$$

$$\begin{aligned}\bar{P}_{\alpha,\bar{\alpha},\gamma,\varepsilon_i} = {} & e^+ \otimes e^+ + \frac{1+\alpha}{2}\, e^0 \otimes e^0 \\ & + \frac{1+\bar{\alpha}}{2}\, e^- \otimes e^- + \frac{e^0 \otimes e^+ + e^+ \otimes e^0 + e^- \otimes e^+ + e^+ \otimes e^-}{2} + e^0 \otimes e^- \\ & + 2\left(e^{\varepsilon_3} S^{++} \otimes S^{--} + e^{-\varepsilon_1} S^{-z} \otimes S^{z+} + \left(e^{\varepsilon_3-\varepsilon_1}\right)\gamma S^{+z} \otimes S^{z-}\right).\end{aligned} \tag{4.18}$$

The operators $P_{\alpha,\bar{\alpha},\gamma,\varepsilon_i}$ at the values $\gamma = \pm 1$, $\varepsilon_i = \pi n_i$ (n_i are integers) just correspond to the permutation operators – the signs $\alpha, \bar{\alpha} = \pm 1$ and γ, e^{ε_i} are responsible to the different gradings of the spaces. Such that the ordinary non-graded case $\alpha = \bar{\alpha} = 1$, $\gamma = 1$, $\varepsilon_i = 0$ corresponds to the permutation operator of the $sl(2)$-invariant spin-1 spaces. The values $\alpha = -1$, $\bar{\alpha} = 1$, $\gamma = 1$, $\varepsilon_i = 0$ correspond to the $osp(1|2)$-invariant case with the fundamental three-dimensional spin-1/2 representation spaces with two even and one odd parity vectors. When $\alpha = -1$, $\bar{\alpha} = 1$, $\gamma = 1$, $e^{\varepsilon_1} = -1$, $e^{\varepsilon_3} = 1$, the corresponding operator, multiplied by a minus sign, is the permutation acting on the spaces with two odd and one even parity vectors. For the three-dimensional solutions of the mentioned algebras and their quantum extensions see the papers [29–33], and it is interesting to mention that taking into account the correspondence between the representations of the quantum algebras $osp_q(1|2)$ and $sl_{i\sqrt{q}}(2)$, the mentioned graded matrices can be obtained from the $sl_t(2)$-invariant matrices at $t = 1, i$. The next summand with $\bar{P}$ (4.18) exists when $x_f \neq 1$ and spoils the mentioned symmetries even at the points $\gamma = 1$, $\varepsilon_i = 0$. We must mention once again, that in the course of the solution we have taken $c_2 \neq \bar{c}_2$, and this choice induces the mentioned Hamiltonian. The solutions with $c_1 \neq \bar{c}_1$ or $c_3 \neq \bar{c}_3$ would bring to slight changes in the structure of the asymmetric part $\bar{P}$, conditioned by the appropriate interchanges of the coefficients in (4.15).

5. Summary

In this paper we have completed the list of the colored R_{22} solutions by multi-parametric free-fermionic solutions. Colored R_{33}-matrices are obtained for the matrices with 15 non-zero entries. The given approach for the solving the multi-parametric YBE, being very simple and clear-cut, gives an opportunity to find colored YBE solutions for higher-dimensional matrices. It is shown, that the number of the possible extra colors, on which the R-matrix can be dependent, is restricted: initially the number of the elementary functions (the matrix elements of $R(\mathbf{u},\mathbf{0})$ taken at the normalization point $\mathbf{0}$) is defined from the number of the non-zero elements, and then the set of the independent relations following from YBE imposed on the elementary functions determines the number of the arbitrary functions (or the number of the possible colors) existing in the solutions. The solutions for YBE (*) $K(\mathbf{u},\mathbf{w},\mathbf{0})=0$ ($K(\mathbf{u},\mathbf{0},\mathbf{w})=0,\ K(\mathbf{0},\mathbf{u},\mathbf{w})=0$) are sufficient to solve the whole set of the YBE, as the taken point $\mathbf{0}$ is chosen arbitrarily.

The discussed multi-parametric R-matrices being the solutions of the Yang–Baxter equations, can have usage and treatment in all areas of the theoretical and mathematical physics, where YBE are involved – integrable models, high energy physics, quantum groups, quantum information theory, statistical physics. We have seen that the four-parametric 4×4-solutions presented in Section 2.3.3 have their interpretation as intertwiner matrices in the theory of the quantum algebra [18] (i.e. all free-fermionic solutions can be presented as $sl_q(2)$-invariant matrices at $q=i$), and the colored parameters are the characteristics of the representations of the quantum algebra. For the cases with higher dimensions also one could expect the existence of respective underlying symmetry. In the framework of the Algebraic Bethe Ansatz the multi-parametric solutions bring to the richer families of integrable Hamiltonians. The cases when there are more than one pair of the spectral parameters with "difference" property (which can be interpreted as "rapidities"), as it was in (2.38), (A.23), are of particular interest.

The all obtained solutions are unitary in the following sense

$$\check{R}(\mathbf{u},\mathbf{w})\check{R}(\mathbf{w},\mathbf{u})\approx\mathbf{I}. \tag{5.19}$$

This relation can be proved for all the cases, using only the symmetry property of the functions $f_i(\mathbf{u},\mathbf{w})$ under the interchange of the variables $\mathbf{u}$ and $\mathbf{w}$, and the compatibility conditions of the Yang–Baxter equations. Particularly, the unitarity relation for the R_{22}-matrix

$$a_1(\mathbf{u},\mathbf{w})a_1(\mathbf{w},\mathbf{u})+d(\mathbf{u},\mathbf{w})d(\mathbf{w},\mathbf{u})=1+b_1(\mathbf{u},\mathbf{w})b_2(\mathbf{w},\mathbf{u}), \tag{5.20}$$

for the free-fermionic solutions just means the free-fermionic property. If to take the usual unitarity condition for the matrices,

$$\check{R}^{+}(\mathbf{u},\mathbf{w})=\check{R}^{-1}(\mathbf{u},\mathbf{w}), \tag{5.21}$$

then we shall have some restrictions on the solutions, but the number of the arbitrary functions does not changed, as the relation (5.21) means

$$a_1(\mathbf{u})^{+}=a_2(\mathbf{u}),\qquad b_1^{+}(\mathbf{u})=-b_2(\mathbf{u}),\qquad d^{+}(\mathbf{u})=-d(\mathbf{u}). \tag{5.22}$$

Here there are five relations on the ten functions $\mathrm{Re}[f_i(\mathbf{u})]$, $\mathrm{Im}[f_i(\mathbf{u})]$ and again we have five elementary functions: $\mathrm{Re}[a_1(\mathbf{u})](=\mathrm{Re}[a_2(\mathbf{u})])$, $\mathrm{Re}[b_1(\mathbf{u})](=-\mathrm{Re}[b_2(\mathbf{u})])$, $\mathrm{Im}[a_1(\mathbf{u})](=-\mathrm{Im}[a_2(\mathbf{u})])$, $\mathrm{Im}[b_1(\mathbf{u})](=\mathrm{Im}[b_2(\mathbf{u})])$ and $\mathrm{Im}[d(\mathbf{u})]$ ($\mathrm{Re}[d(\mathbf{u})]=0$), with further (in general three) relations on them.

Note

We could investigate in the same manner the cases with the conditions $f_1=0,\ f_2\neq 0$ or $f_2=0,\ f_1\neq 0$ with $f=b,d$ (the situations with $f=a,c$ would spoil the condition (2.3) and bring to rather trivial solutions). As we have learnt from [17,18] such cases can include separate solutions, together with the limit cases of the general solutions which one can obtain after taking the appropriate limits ($f_i\to 0$, $i=1$ or $i=2$), and however all the solutions have functional dependence (i.e. existence of the definite number of arbitrary functions, or in the particular cases, elliptic, trigonometric and rational future of the solutions) similar to the case of general solutions.

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Appendix A. The independent equations in the set of YBE with the conditions $d(\mathbf{u}) \neq 0$ and the derivation of a solution at $a_1'(0) = -a_2'(0)$

Here we discuss the set of the YBE for the case with the conditions $b_1(\mathbf{u})a_1(\mathbf{u}) = b_2(\mathbf{u})a_2(\mathbf{u})$, $a_1'(\mathbf{0}) = -a_2'(\mathbf{0})$. As in the general case, here there are six independent equations in the set of YBE. After performing the following notations

$$f_1(\mathbf{u},\mathbf{v}) = f_{12}, \qquad f_2(\mathbf{u},\mathbf{v}) = f_{x12}, \qquad f_1(\mathbf{u},\mathbf{w}) = f_{13}, \qquad f_2(\mathbf{u},\mathbf{v}) = f_{x12},$$
$$f_1(\mathbf{v},\mathbf{w}) = f_{23}, \qquad f_2(\mathbf{v},\mathbf{w}) = f_{x23}, \qquad f = a, b, d, \tag{A.1}$$

the independent equations look like as

$$a_{12}a_{23} - a_{13} - b_{23}b_{x12} + a_{x13}d_{12}d_{23} = 0, \tag{A.2}$$
$$a_{12}d_{13} - a_{x12}d_{23} - a_{13}a_{23}d_{12} + b_{x13}b_{x23}d_{12} = 0, \tag{A.3}$$
$$a_{12}b_{13} - a_{13}b_{12} - b_{23} + b_{x23}d_{12}d_{13} = 0, \tag{A.4}$$
$$a_{23}b_{x13} - b_{x12} - a_{13}b_{x23} + b_{12}d_{13}d_{23} = 0, \tag{A.5}$$
$$a_{23}d_{13} - a_{x23}d_{12} - a_{12}a_{13}d_{23} + b_{12}b_{13}d_{23} = 0, \tag{A.6}$$
$$b_{x13}d_{12} - b_{13}d_{23} + a_{12}b_{23}d_{13} - a_{23}b_{x12}d_{13} = 0. \tag{A.7}$$

The remaining equations can be obtained from these ones doing some changes of the arguments, taking into account that

$$a_1(\mathbf{u},\mathbf{w}) = a_2(\mathbf{w},\mathbf{u}), \qquad b_i(\mathbf{u},\mathbf{w}) = -b_i(\mathbf{u},\mathbf{w}), \qquad d(\mathbf{u},\mathbf{w}) = -d(\mathbf{w},\mathbf{u}). \tag{A.8}$$

Let us prove that under the mentioned conditions the relations (2.43), (2.44), (2.45) imposed on the functions $f_i(\mathbf{u})$, $f = a, b, d$ are enough to solve the whole set of YBE. The equations appear to be large and rather complicated and for simplifying the evaluation process we could suggest the following reparameterizations. Let us introduce new functions $f(\mathbf{u})$ and $g(\mathbf{u})$, such that

$$a_1(\mathbf{u}) = g(\mathbf{u})a_2(\mathbf{u}), \qquad b_1(\mathbf{u}) = f(\mathbf{u})a_2(\mathbf{u}). \tag{A.9}$$

We can parameterize them by means of the function $d(\mathbf{u})$ and two constants f_0, x_f

$$f(\mathbf{u}) = f_0\left(f_g(\mathbf{u}) - \sqrt{f_g(\mathbf{u})^2 - \frac{1}{f_0^2}}\right), \qquad g(\mathbf{u}) = \frac{2d(\mathbf{u})}{1+d(\mathbf{u})^2}\big(x_f + f_g(\mathbf{u})\big), \tag{A.10}$$

$$f_g(\mathbf{u}) = \sqrt{x_f^2 + \frac{(1+d(\mathbf{u})^2)^2}{4d(\mathbf{u})^2}}, \qquad a_2(\mathbf{u}) = \sqrt{\frac{1+d(\mathbf{u})^2}{g(u)(1+f(\mathbf{u})^2)}}. \tag{A.11}$$

The two parametric functions followed from (2.30) can be written in rather compact formulas

$$a_1(\mathbf{u},\mathbf{w}) = a_2(\mathbf{u})a_2(\mathbf{w})\big[1+f(\mathbf{u})f(\mathbf{w})\big]\mathcal{A}_1(\mathbf{u},\mathbf{w}) \tag{A.12}$$

$$a_2(\mathbf{u},\mathbf{w}) = a_2(\mathbf{u})a_2(\mathbf{w})\big[1+f(\mathbf{u})f(\mathbf{w})\big]A_2(\mathbf{u},\mathbf{w}) \tag{A.13}$$

$$b_1(\mathbf{u},\mathbf{w}) = a_2(\mathbf{u})a_2(\mathbf{w})\big[f(\mathbf{u})-f(\mathbf{w})\big]\mathcal{B}_1(\mathbf{u},\mathbf{w}) \tag{A.14}$$

$$b_2(\mathbf{u},\mathbf{w}) = a_2(\mathbf{u})a_2(\mathbf{w})\big[f(\mathbf{u})-f(\mathbf{w})\big]\mathcal{B}_1(\mathbf{u},\mathbf{w}) \tag{A.15}$$

$$d(\mathbf{u},\mathbf{w}) = \frac{f(\mathbf{u})-f(\mathbf{w})}{f(\mathbf{u})+f(\mathbf{w})}\mathcal{D}(\mathbf{u},\mathbf{w}). \tag{A.16}$$

Here we have introduced the following functions, which can be parameterized only by the functions $g(\mathbf{u})$, $d(\mathbf{u})$ (from Eqs. (A.10) it follows $f_g(\mathbf{u}) = g(\mathbf{u})\frac{(1+d(\mathbf{u})^2)}{2d(\mathbf{u})} - x_f$)

$$\mathcal{A}_1(\mathbf{u},\mathbf{w}) = \frac{d(\mathbf{w})+d(\mathbf{u})g(\mathbf{u})g(\mathbf{w})}{2d(\mathbf{u})d(\mathbf{w})[f_g(\mathbf{u})+f_g(\mathbf{w})]}, \qquad A_2(\mathbf{u},\mathbf{w}) = A_1(\mathbf{w},\mathbf{u}) \tag{A.17}$$

$$\mathcal{B}_1(\mathbf{u},\mathbf{w}) = \frac{1+d(\mathbf{u})d(\mathbf{w})g(\mathbf{u})g(\mathbf{w})}{1-d(\mathbf{u})^2d(\mathbf{w})^2}, \qquad \mathcal{B}_2(\mathbf{u},\mathbf{w}) = \frac{d(\mathbf{u})d(\mathbf{w})+g(\mathbf{u})g(\mathbf{w})}{1-d(\mathbf{u})^2d(\mathbf{w})^2} \tag{A.18}$$

$$\mathcal{D}(\mathbf{u},\mathbf{w}) = \frac{(d(\mathbf{u})-d(\mathbf{w}))(1-g(\mathbf{u})g(\mathbf{w}))}{(1+d(\mathbf{u})d(\mathbf{w}))(g(\mathbf{u})-g(\mathbf{w}))} \tag{A.19}$$

We see that the functions factorize into two parts which contain the function $f(\mathbf{u})$ and the functions $d(\mathbf{u})$, $g(\mathbf{u})$. In the equations we can expand the relations on the series in terms of the constant f_0 and the function-factors $f_{x1} = \sqrt{f_g(\mathbf{u})^2 - \frac{1}{f_0^2}}$, $f_{x2} = \sqrt{f_g(\mathbf{v})^2 - \frac{1}{f_0^2}}$, $f_{x3} = \sqrt{f_g(\mathbf{w})^2 - \frac{1}{f_0^2}}$, as they meet only in the functions $f(\mathbf{u})$, $f(\mathbf{v})$, $f(\mathbf{w})$. Also one can notify that the functions $a_2(\mathbf{u})$ either have been factorized from the equations or met there in the quadratic form $a_2(\mathbf{u})^2 = \frac{a_f(\mathbf{u})}{1+f(\mathbf{u})^2}$, $a_f(\mathbf{u}) = \frac{1+d(\mathbf{u})^2}{g(\mathbf{u})}$, which allows us to escape the double square roots in the equations.

As example, let us represent the investigation of the first equation (A.2). After the mentioned expansion we find, that there are only two following equations, which one has to prove and which can be done after some not so complicated calculations.

$$A_1(\mathbf{u},\mathbf{w}) - a_f(\mathbf{v})B_1(\mathbf{v},\mathbf{w})B_2(\mathbf{u},\mathbf{w}) + D(\mathbf{u},\mathbf{v})D(\mathbf{v},\mathbf{w})A_2(\mathbf{u},\mathbf{w}) = 0, \tag{A.20}$$

$$\begin{aligned}&2A_1(\mathbf{u},\mathbf{w})f(\mathbf{v})\big[f_g(\mathbf{u})+f_g(\mathbf{w})\big] - a_f(\mathbf{u})\big(B_1(\mathbf{v},\mathbf{w})B_2(\mathbf{u},\mathbf{v})\big[f_g(\mathbf{u})-f_g(\mathbf{v})\big]\\ &\quad\times\big[f_g(\mathbf{v})-f_g(\mathbf{w})\big] + A_1(\mathbf{u},\mathbf{v})A_1(v,\mathbf{u})\big[f_g(\mathbf{u})+f_g(\mathbf{v})\big]\big[f_g(\mathbf{v})+f_g(\mathbf{w})\big]\big) = 0.\end{aligned} \tag{A.21}$$

A.1. The main 4×4 *solutions to YBE obtained heretofore*

Here we are presenting in the apparent matrix form the already known solutions, observed and classified in [5,13,14] and two-parametric solutions in [17,18].

The eight-vertex solution with elliptic parameterization is [5]

$$R_{xyz}(u) = \begin{pmatrix} \frac{\mathrm{sn}[u+\lambda,\,k]}{\mathrm{sn}[\lambda,\,k]} & 0 & 0 & e^{\gamma/2}k\ \mathrm{sn}[\lambda+u,k]\,\mathrm{sn}[u,k] \\ 0 & \frac{\mathrm{sn}[u,\,k]}{\mathrm{sn}[\lambda,\,k]} & 1 & 0 \\ 0 & 1 & \frac{\mathrm{sn}[u,\,k]}{\mathrm{sn}[\lambda,\,k]} & 0 \\ e^{-\gamma/2}k\ \mathrm{sn}[\lambda+u,k]\,\mathrm{sn}[u,k] & 0 & 0 & \frac{\mathrm{sn}[u+\lambda,\,k]}{\mathrm{sn}[\lambda,\,k]} \end{pmatrix}. \tag{A.22}$$

The following relation takes place $\frac{a^2(u)+b^2(u)-c^2(u)-d^2(u)}{a(u)b(u)} = 2\,\mathrm{cn}[\lambda,k]\mathrm{dn}[\lambda,k]$.

Free fermionic case corresponds to the relation $\mathrm{cn}[\lambda,k]\mathrm{dn}[\lambda,k] = 0$. The XY-model corresponds to $\lambda = K(k)$.

All other solutions presented below have free-fermionic property.

The non-homogeneous solution $a_1(\mathbf{u}) = a_2(\mathbf{u})$, $b_1(\mathbf{u}) = -b_2(\mathbf{u})$ corresponds to [17]

$$\tilde{R}(u;v) = \begin{pmatrix} \cosh[u] & 0 & 0 & e^{\gamma/2}\sin[v] \\ 0 & \sinh[u] & \cos[v] & 0 \\ 0 & \cos[v] & -\sinh[u] & 0 \\ e^{-\gamma/2}\sin[v] & 0 & 0 & \cosh[u] \end{pmatrix}. \tag{A.23}$$

and satisfies to the YB equations (1.5).

The non-homogeneous one-parametric solution $a_1(\mathbf{u}) \neq a_2(\mathbf{u})$, $b_1(\mathbf{u}) = b_2(\mathbf{u})$ corresponds to [17]

$$R_I(u) = \begin{pmatrix} \frac{\mathrm{dn}[u,k]}{\mathrm{cn}[u,k]} \pm \frac{\mathrm{dn}[u_0,k]\,\mathrm{sn}[u,k]}{\mathrm{sn}[u_0,k]} & 0 & 0 & e^{\gamma/2}\frac{\mathrm{dn}[u,k]\,\mathrm{sn}[u,k]}{\mathrm{cn}[u,k]} \\ 0 & \frac{\mathrm{sn}[u,k]}{\mathrm{sn}[u_0,k]} & 1 & 0 \\ 0 & 1 & \frac{\mathrm{sn}[u,k]}{\mathrm{sn}[u_0,k]} & 0 \\ e^{-\gamma/2}\frac{\mathrm{dn}[u,k]\,\mathrm{sn}[u,k]}{\mathrm{cn}[u,k]} & 0 & 0 & \frac{\mathrm{dn}[u,k]}{\mathrm{cn}[u,k]} \mp \frac{\mathrm{dn}[u_0,k]\,\mathrm{sn}[u,k]}{\mathrm{sn}[u_0,k]} \end{pmatrix}. \tag{A.24}$$

If $u_0 = K(k)$, where K is the complete elliptic integral of the first kind with the module k, then the matrix $R_I(u)$ corresponds exactly to the R-matrix, which we have found in [10] for $2d$ Ising Model. After so-called "transformations of the first degree" of the elliptic functions [12], this solution at $u_0 = K(k)$ corresponds to the XY-model matrix.

The three-parametric (colored) solution in [14,13]. Defining the function $\mathrm{e[u,k]}$ as $\mathrm{e[u,k]} = \mathrm{cn[u,k]} + i\,\mathrm{sn[u,k]}$ the matrix is written as

$$R(\mathrm{u};p,q) = \begin{pmatrix} 1-\mathrm{e[u,k]}pq & 0 & 0 & \frac{\mathrm{k}\sqrt{(1-p^2)(1-q^2)}(1+\mathrm{e[u,k]})\,\mathrm{sn}[\frac{\mathrm{u}}{2},\mathrm{k}]}{2} \\ 0 & q-p\mathrm{e[u,k]} & \frac{i\sqrt{(1-p^2)(1-q^2)}(1-\mathrm{e[u,k]})}{2\,\mathrm{sn}[\frac{\mathrm{u}}{2},\mathrm{k}]} & 0 \\ 0 & \frac{i\sqrt{(1-p^2)(1-q^2)}(1-\mathrm{e[u,k]})}{2\,\mathrm{sn}[\frac{\mathrm{u}}{2},\mathrm{k}]} & p-q\mathrm{e[u,k]} & 0 \\ \frac{\mathrm{k}\sqrt{(1-p^2)(1-q^2)}(1+\mathrm{e[u,k]})\,\mathrm{sn}[\frac{\mathrm{u}}{2},\mathrm{k}]}{2} & 0 & 0 & \mathrm{e[u,k]}-pq \end{pmatrix}. \tag{A.25}$$

The second solution given in Subsection 2.3.2 corresponds to this case after transformation $\mathrm{k} \to 1/k = i/x_f$. Taking the symmetric parameterization $p = \mathrm{e}[\psi_1,\mathrm{k}]$, $q = \mathrm{e}[\psi_2,\mathrm{k}]$ one can write the matrix as function $R(\mathrm{u};\psi_1,\psi_2)$.

$sl_q(2)$-invariant solution defined on the two-dimensional cyclic irreps at $q = i$. The $sl_q(2)$-invariant 4×4 R-matrices at $q = i$ have the free-fermionic property: the solutions defined on the spin-irreps correspond to the trigonometric XX limit of the R_{xyz}-matrix (A.22) when $d_i = 0$, and the first observed solution [20] on the cyclic (semi-cyclic, nilpotent) irreps corresponds to the same $d_i = 0$ limit of the inhomogeneous matrix (A.24). The solutions defined on the cyclic irreps with special degenerated cases are described in [18] (Sections 4.3, 4.4), where $c_i \cosh\varepsilon_j = c_j \cosh\varepsilon_i = 0$, with $c_{i,j}$, $\exp\{\varepsilon_{i,j}\}$ being the eigenvalues of the Casimir operators c and k^2 from the extended center of the quantum algebra. Besides of the special trigonometric limits, now with $d_i \neq 0$, of the matrices (A.22), (A.24), (A.23), (A.25), the solutions presented

there include also general solutions with two constants which do not admit difference property. The mentioned solutions can coincide with the solutions obtained here in Subsection 2.3.3 having two arbitrary functions and two arbitrary constants, after fixing the arbitrary functions (say $d(\mathbf{u})$, $f_g(\mathbf{u})$) to have the following dependence on the exponential function e^ε and the arbitrary function $\bar{h}(\varepsilon)$ established in [18]

$$d(\mathbf{u}) = \bar{h}(\varepsilon), \tag{A.26}$$

$$f_g(\mathbf{u}) = i\frac{[\bar{f}(\varepsilon)e^\varepsilon - 1] + \bar{h}(\varepsilon)[\bar{f}(\varepsilon) - g_0 e^\varepsilon]}{[e^\varepsilon + \bar{f}(\varepsilon)] - \bar{h}(\varepsilon)[e^\varepsilon \bar{f}(\varepsilon) + g_0]}, \tag{A.27}$$

$$2h_0\bar{f}(\varepsilon)\bar{h}(\varepsilon) = 1 - \bar{f}(\varepsilon)^2 + \bar{h}(\varepsilon)^2\big[g_0^2 - \bar{f}(\varepsilon)^2\big]. \tag{A.28}$$

The function $\bar{f}(\varepsilon)$ is related to ε and $\bar{h}(\varepsilon)$ by means of the last relation, h_0 and g_0 are arbitrary numbers [18], which can be expressed by the constant parameters f_0 and x_f.

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